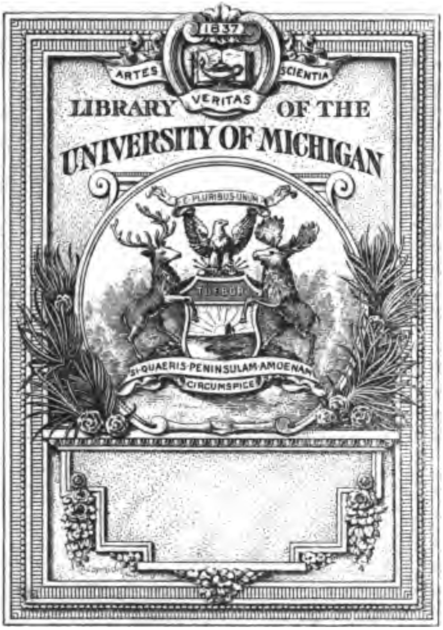
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THE PRINCIPLES OF MECHANICS

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THE PRINCIPLES OF MECHANICS

AN ELEMENTARY EXPOSITION

FOR STUDENTS OF PHYSICS

BY

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PART I

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PREFACE

THE material contained in these chapters has taken on its present form gradually, by a process of recasting and sifting. The ideas guiding that process have been three: first, to select the subject-matter with close reference to the needs of college students; second, to bring the instruction into adjustment with the actual stage of their training; and, third, to aim continually at treating Mechanics as a system of organized thought, having a clearly recognizable culture value. The result has been maturing under the suggestive influences of repeated presentation to a class of junior grade in the University who have brought to their task a working knowledge of calculus and a good groundwork of experimental physics. The distinctive traits in this attempt to set forth connectedly the conceptions and methods of the science are reproduced from the informal lectures of the class-room, as delivered for several years past. Advantage has been taken, however, of the larger opportunity in writing for publication, to make at some points a completer and more systematic formulation of fundamental doctrine. The seventh chapter has been incorporated here from another course of instruction, in part, at least, because its problems afford exactly the illustration needed of the difference between mathematical and physical forces. The scope of this first instalment does not extend beyond uniplanar motion;

but a continuation is provided for, which is to include the unconstrained motion of a rigid body and those fully generalized principles of Mechanics whose absence from this first part will be at once noticed.

The following chapters may be arranged in two groups, and the first of these (Chapters I–V) is devoted continuously to building up knowledge of the necessary conceptions, and unfolding their content. It is believed that such a plan will foster in students who use the book some quality of precision and method in statement, while it accustoms them to follow a sustained argument. If results like these are not attained by those who cultivate the science of Mechanics, the benefits of their study are but half reaped. The ninth and tenth chapters are given up mainly to auxiliary discussions on Centre of Mass, Moment of Inertia, and Dimensions; leaving three chapters in that group (Chapters VI–VIII) for the usual range of applications. This has been covered with a fair degree of thoroughness; and the problems constitute at the same time a collection of varied illustrations, by means of which the general principles can be elucidated and enforced.

The arrangement adopted does not indicate any intention that the student should be led to attack the chapters in the order of their numbering. In learning a foreign language the study of its grammar should proceed *pari passu* with that of specimens from its literature; and just so here. Mechanics has a language of conceptions as well as a grammar of principles; and the problems of Mechanics are like a literature to exhibit that language in use. The effectiveness of the instruction will hinge upon discretion in the selection of problems at the right moment, to fortify the knowledge already gained and prepare the

ground for new advance. Those who are called upon to instruct will know how to intermingle judiciously theory with graded application.

There is some diffused feeling that students seldom get at the heart of our subject. Professor Klein has remarked recently that comprehension of the ideas in Mechanics is rarely attained; and with sound judgment he proceeds to locate the cause of failure in too exclusive attention given to the analytic deduction of general equations. His criticism is made for his own country, but it contains profound truth that applies outside the boundaries of Germany. Yet my personal experience confirms the belief that students can reach Professor Klein's goal of gaining "not merely a knowledge of Mechanics, but a sense (or feeling) of its truths." Only they must approach the science through its genetic relations with physics, and not through its formal similarities to mathematics. And, finally, they must be painstaking in understanding its elements before they proceed to its more ambitious generalizations. A deliberate purpose of setting the feet of students on the road to that success is announced by the sub-title of this book.

F. S.

UNIVERSITY OF CALIFORNIA,
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THE PRINCIPLES OF MECHANICS

INTRODUCTORY

The science of Mechanics is concerned with the physical phenomena involved in motion. Its material is to be found wherever there are motions observable in larger bodies, or attributed to their minute parts by inferences drawn from conditions that are more directly accessible to our senses. The problem of Mechanics, as formulated a quarter-century ago by one leading thinker, is: "To describe the motions occurring in nature completely and in the simplest terms." In order that such description may approach completeness, it must become quantitatively definite; and if the terms used are the simplest available, there must have been searching scrutiny of the cases presented in experience, with invention of the conceptions that seemed necessary and discriminating choice among those offered for acceptance.

First, then, the study of scientific history makes it evident that processes of selection, clarification, and extension have been going forward with the ideas at the foundation of Mechanics. These tendencies are still continuously operative to modify — perhaps subtly and insensibly — the tenets of our doctrine; the thought of the science to-day is not as Newton left it, however minute the formal changes may appear to be. There is a field of fruitful suggestion here for serious students of the subject. Secondly, we find that Mechanics is preëminently a science of measurements. Moreover, a large proportion of the exacter quantitative determinations in what

are nominally other branches of physics are rendered possible only by a close alliance with Mechanics, and liberal use of fundamental ideas furnished by it. In a very true sense, this subject is the threshold across which entry is gained to advanced study on many other important lines.

The closely knit relations with other departments of physics are grounded in the circumstance that Mechanics deals distinctively with those qualities in bodies which are universal, and inseparable in our experience from the notion of *matter*. Whatever specialized phenomena of light, magnetism, and the like may show themselves as consequences of more occasional properties in bodies, they are always superposed upon those assigned particularly to Mechanics. Through all of its transformations, energy carries with it a suggestion of the mechanical point of view.

The calculus notation will be introduced at once in beginning our subject; but observe that no quantitative science stands committed to the procedure of mathematics in reasoning, as well as in operations, because it utilizes the mathematical symbols and processes. The natural sciences build upon experimental data, and do not content themselves with definition and postulate apart from appeal to fact.

In the first two chapters, however, we shall be occupied with conceptions — Velocity and Acceleration — that rest entirely upon a mathematical basis. They are connected with the study of Mechanics as a preliminary, for convenience solely. There is an important and extensive branch of mathematics that deals with motions of points and systems of points. It is known as Kinematics; velocity and acceleration, as attached in thought to points and not bodies, are kinematical quantities. They are treated of here, in view of their future usefulness for our main purpose. If Mechanics is taken to include Kinematics also, as it frequently is, that part

of the science which is physical and not geometrical must be specially distinguished. It is designated as Dynamics. The study of Mechanics proper, or Dynamics, begins with the third chapter. The point should be watched at which the transition is there made by introducing experimental results into the framework of our science. The problems of Dynamics are further classified, according as they deal with states of rest or motion in bodies. The corresponding subdivisions of Dynamics are: Kinetics, comprising the ideas prominent in cases of motion; Statics, the principles in forms that apply to rest.

Three pillars support the superstructure of Dynamics: (I) The conception of Inertia (§§ 40, 42); (II) the Equation of Motion (§§ 43, 46); (III) the postulate that all forces are in final analysis internal stresses (§ 47). These constitute the Laws of Motion, which it is Newton's achievement — perhaps the greatest among his fruitful labors — to have deciphered in the complex phenomena. The three laws as he formulated them are given in the Appendix.

KINEMATICAL

CHAPTER I

SPEED AND VELOCITY

1. Position is specified in terms of some system of coördinates; and involves choice of origin from which to measure lengths, as well as reference-axes from which to measure angles.

In a large proportion of the problems that offer themselves in Mechanics, it proves sufficient and most convenient to choose origin and reference-axes in permanent relation to the earth. For example, vertical and horizontal lines at a given place, or the earth's centre, together with the polar axis, and definite diameters of the equatorial plane. But such reference to the earth is not a necessity. For astronomical purposes, for instance, the origin may be chosen at the centre of the sun; or the position of the sun itself may be assigned with reference to the general arrangement of "fixed stars." In any case the reference-system may be selected at will, according to convenience.

2. A point is said to be **at rest** or **fixed in relation to any reference-system,** if its position is not changing. It is said to be **in motion,** if any element of its position is changing.

Let Q be any point, and O the origin. The position of Q may be given by means of the length OQ, and the angles which that line makes with any two reference-axes drawn through O. If none of these elements is varying, Q is at rest; otherwise it is in motion.

These relations of rest or motion may be either instantaneous or permanent. Thus in an ordinary clock pendulum, all its points are at rest relatively to the earth, at each end of the swing. There is permanent rest with reference to the same elements, if the pendulum hangs in the position of equilibrium. Common usage connects the word "fixed" with permanent rest.

Position is essentially a relative idea. The constancy or variableness of position must be relative also. Consequently no definite statement is possible concerning the rest or motion of a point, until a reference-system has been determined upon. There may be rest for a given point relatively to a system chosen in the earth, and motion relatively to another chosen in the sun; rest relative to the deck of a vessel, consistent with motion relative to the shore. Therefore, when the position of a point is spoken of, or its state of rest or motion, the idea, "with respect to the chosen reference-system," is in every case understood to be implied.

The term "reference-system" is not synonymous with "system of coördinates." The vital distinction lies in the fact that the reference-system, by the very conception of it, must be regarded as fixed, so long as it is retained for a basis; while coördinate lines may move. This is apparent in connection with plane polar coördinates. The reference-system comprises the pole, and the line from which the polar angle is measured. The coördinates are the polar angle and the radius-vector; the latter is a moving line. This distinction may always be maintained with profit; it becomes prominent and important in the advanced stages of the subject.

3. Time is introduced in Mechanics as a quantity continuously varying, whose uniform flow is in the direction of larger positive values. The zero-point of time may be arbitrarily

taken; it is often conveniently assumed at the beginning of some particular discussion or problem. Then "initial conditions" correspond to zero of time, and negative values of time will occur as exceptions only, when the investigation is as it were prolonged backward. This conception of time as a variable in a state of continuous and uniform flow, in terms of which order and succession are assignable, was the basis of the "Fluxion" as invented by Newton. It evidently suggests and determines the general use of time as the independent variable in problems of Mechanics, where operations of calculus occur.

Numerical measure of time will ordinarily be made in **mean solar seconds.** A mean solar day is the average value for the tropical year, of the interval between successive transits of the sun across the same meridian. There are 86,400 mean solar seconds in a mean solar day. This time-unit will be called **second,** for brevity.

4. The group of positions which a point may occupy, consistently with the conditions or constraints imposed upon its motion, is called its **path.** This agrees with the conception of locus in geometry, except that the idea of positions as successive in time is prominent in path, regarded as traced or described by the point. The term "path" may often be applied conveniently to the complete locus, even though the point does not actually traverse all of it. It is a postulate of Physics that matter, the assumed basis of physical phenomena, has continuous existence. Therefore the points of any individual portion of matter, or body, will necessarily describe continuous paths. The path of a point is evidently another relative conception; it depends for its form and situation upon the choice of reference-system. Take A as origin (Fig. 1), draw the reference-axis AB, and let the circle, centre O, roll (without slipping) upon AB, in the plane of the diagram. The path

described by P, a point of its circumference, is a cycloid. If the reference-system is: Origin O, reference-axis OB' always parallel to AB, the path is a circle. If, finally, O be origin, and a definite radius of the circle be reference-axis, the path of P is a point; *i.e.* it is fixed relatively to a reference-system so chosen.

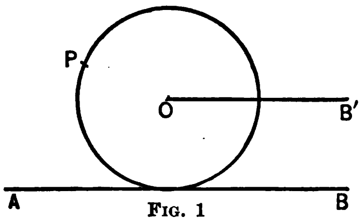


FIG. 1

The coördinates of a point are fundamentally regarded as functions of time, and this relation may be expressed by simultaneous equations involving coördinates and time. Elimination of time from such simultaneous equations gives the equation of the path. Thus if X and Y be rectangular reference-axes, $x = at$, $y = bt$ represent a straight line; $x = a\cos(kt)$, $y = a\sin(kt)$, a circle; if a, b, k, are constants. When a path is to be plotted, the reference-axes are drawn, and the values of the coördinates determined, which give for any instant considered the position of the point.

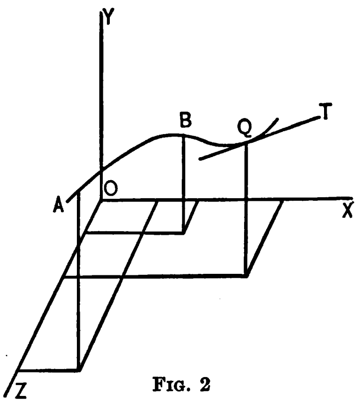


FIG. 2

5. Let ABQ (Fig. 2) be any curve, which is the path of a point Q for a given reference-system. Let A be any fixed point in ABQ, and s a coördinate measured from A along the path, positively in the general direction ABQ. If Δt is the time occupied by Q in passing over any length Δs, the limiting ratio of the length to the time is defined as the

speed of Q in its path, at the instant from which Δt is reckoned or the point from which Δs is measured. The speed is therefore expressed at each instant by the corresponding value of $\frac{ds}{dt}$. It is the rate at which s is changing per unit time, or the "time-rate" of s.

Plus or minus sign is attached to speed, but not the general idea of direction. The sign of the coefficient $\frac{ds}{dt}$ is determined by the ordinary rule of calculus. The change in t being positive by convention, positive change in s is the condition for positive speed; and negative change in s for negative speed. Therefore a point has positive or negative speed, according as it is moving in the positive direction along its path, or the reverse. The sign and value of speed are independent of the particular point in the curve from which s is measured, so long as that point is fixed, and the positive direction is retained. But the sign of speed is changed by a different choice of positive direction in the path.

For the conditions as stated, Q has positive speed while moving in the general direction from A toward C; but the speed would be negative, for the same actual displacement of Q, if s were measured positively in the direction CBA.

If equal changes occur in s during equal intervals of time, no matter where those intervals begin, speed is constant; otherwise it is variable. In repeated or symmetrical motions, like those of a clock pendulum relatively to the earth's surface, *particular* choice of equal time-intervals may show equal values of length passed over; but this is not sufficient to assure constancy in speed. In the case of really constant speed, its sign and value are given by *any* such quotient as $\frac{s'' - s'}{t'}$, if s' and s'' are the values of s at the beginning and end respectively of the interval whose duration is t'. When the

speed is variable, such a quotient shows the **average speed** for the interval.

The idea of speed, then, deals with the "rapidity," or "intensity," or "magnitude" of the motion. All these terms are found current. Numerical values of speed will depend upon the unit of length adopted, as well as the unit of time. In scientific work, and in many technical lines that are closely connected with science, the length-unit is the centimeter. The international standard meter (kept at Sèvres, near Paris), when at a temperature of zero centigrade, represents one hundred centimeters.

Speed will ordinarily be expressed in centimeters per second $\left(\frac{\text{cm.}}{\text{sec.}}\right)$. But other combinations may be used; such as feet per second, meters per second, miles or kilometers per hour. The meaning to be attached to the numerical measure when speed is variable is explained in § 6.

6. The term **velocity** is used for the idea which includes both rapidity and direction of motion. As related to the path, the velocity of a point at any instant involves the speed, associated with the direction of. the corresponding tangent, drawn in the general direction of the motion. This will be spoken of as the "forward" direction in the tangent. Thus QT (Fig. 2) is the direction of the velocity at Q, for motion from A toward C. Velocity, therefore, shows the idea of **directed magnitude** or **vector quantity** in full development. It may be represented in both essential elements by a line drawn in its direction, on which the speed is laid off to any chosen scale. When such vectors occur in diagrams, letters should be written and read in the order which indicates the direction. So QT above, and not TQ. Quantities that have magnitude simply, and not direction, are **scalars**.

Velocity in the path will be denoted by v, and the same letter will be used for speed. A similar double use of one symbol is common in connection with other vector quantities to be introduced later. Where the context makes it desirable to emphasize the quantity as a vector, the mark ($\frown$) will be placed above the symbol; thus $\overset{\frown}{v}$ means velocity and not speed. The velocities of points will not be spoken of as equal, unless they are in the same direction along parallel lines, and the speeds are equal.

Velocity is **constant**, only so long as both the **direction** and the **speed** of the motion are maintained unchanged. While velocity is constant, therefore, the path must be a straight line and must be traversed with constant speed. In the general case a point will move in a curved path with variable speed. Then, in accordance with the natural view of differential coefficients as related to curves in such cases, the following statement may be made: The velocity at any instant is to be understood as giving the direction in which the point would move, and the length that would be passed over in the next unit of time, if the law of change for direction and speed were at that instant interrupted.

The data of a problem may be such as to suggest direct treatment by means of the ideas of path and velocity developed above. But this method of attack is often less convenient than one which expresses the motion in terms of elements connected with some coördinate-system, so that velocity, path, etc., can be calculated when those elements are given — or *vice versa*. For the present purpose, oblique, rectangular, and polar coördinates will need to be considered. The requisite relations will be established for these in succession.

7. Let Q move with constant velocity v in the plane of the diagram, O with OX and OY being the reference-system

(Fig. 3). The constant angle between v and OX is α; that between OX and OY is i. Let Q be the position at time t; Q' at time $t+1$. Then QQ' represents v. Draw parallels to OX and OY through both Q and Q'. Then $Q_1Q_1' = QA$, and $Q_2Q_2' = QB$, represent the velocities v_x and v_y of the (oblique) projections of Q upon the axes. Hence in this case the velocities of the projections of Q are the projections of the velocity of Q upon the axes; so that v_x and v_y may also denote the projected velocity of Q.

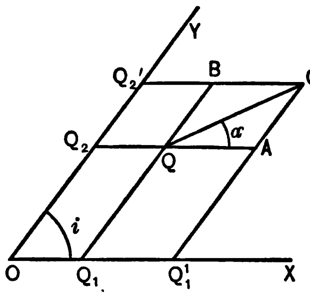


FIG. 3

But this result can be at once extended to cover a more general case. (1) It is seen to apply to three oblique projections in space, Q_1, Q_2, Q_3, and their velocities v_x, v_y, v_z. (2) While the diagram shows the path of Q and the relations that are permanent, for constant velocity, it applies equally to the instantaneous conditions for variable velocity.

If the path is parallel to one of the reference-planes, say XY, the analytical expressions of relation are

$$v^2 = v_x^2 + v_y^2 + 2\,v_xv_y \cos i; \;\; v_y = v\,\frac{\sin\alpha}{\sin i}. \tag{1}$$

The first of these serves to express the speed of v; the second to determine its direction by calculating α, v_x and v_y being supposed given. Or v_x and v_y can be calculated from equations of the second type, the other quantities being given. We may write $v_x = \frac{dx}{dt}$; $v_y = \frac{dy}{dt}$; since these are the speeds of Q_1 and Q_2 in moving along OX and OY respectively. Also $v_z = \frac{dz}{dt}$ if the motion of a third projection is considered. The signs of these

derivatives will follow the rule already formulated with regard to v (§ 5).

If it be desired to extend equations (1) to cover the case of three projections, we may write,

$$v^2 = v_1^2 + v_z^2 + 2\, v_1 v_z \cos\beta\,;\ v_z = v\,\frac{\sin\delta}{\sin\beta}. \qquad (2)$$

Here v_1 represents the result of combining v_x and v_y according to equation (1); β denotes the angle between v_z and v_1; δ the angle between v and v_1.

The velocity of a point in its path is called **tangential** or **resultant velocity**, in order to distinguish it from the velocities of the projections; the latter being known as **component velocities.** Since the motion of a point is in general accompanied by simultaneous motion of its three projections, and the motions of point and projections mutually determine each other, the resultant velocity is said to be equivalent to the three coexistent components. A moving point is often spoken of as having several velocities at the same time. The phrase is convenient as referring to the partial aspects of the motion presented by the component velocities. A point has only one velocity at a given instant, however, in the sense previously defined, provided the reference-system is retained.

8. When the resultant velocity is known, the components are found by projection (resolution; decomposition). Resultant velocity is determined from the components by combining projections (composition). The graphical relations for both directions of the process are compactly shown in Fig. 4. AB represents a resultant velocity v, and is the diagonal of a parallelopiped whose edges drawn from A are parallel to the reference-axes. AE, AF, AD, the oblique projections of AB, represent a set of components v_x, v_y, v_z. All parallel edges

being equal in the figure, any edge parallel to AE may represent a component equal to that laid off on AE itself, like direction in the parallels being maintained; for example, DC, FG. Similarly for AF and AD.

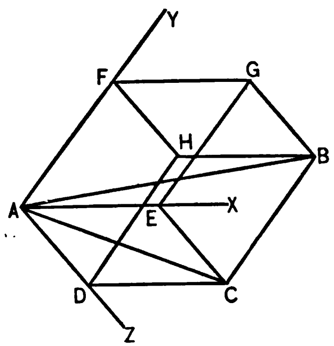


FIG. 4

The resultant velocity can be found by two types of construction. First, find the (partial) resultant of any two components. This will be represented by a diagonal drawn from A in one of the three faces meeting at that point. Then combine this with the remaining component. For example, the combination of v_x and v_z gives the velocity represented by the diagonal AC; and this combined with v_y gives v. Secondly, trace any broken line along three edges, from A to B. This amounts to arranging the vectors v_x, v_y, v_z, in any continuous sequence. Both constructions show the result to be independent of the order in which the components are used.

When applied to two components, the foregoing constructions embody the ideas of the **parallelogram**, and the **triangle**, of **velocities**, which may be stated as follows: —

(1) Adjacent sides of a parallelogram, and the diagonal drawn through their intersection, may represent two component velocities and their resultant, respectively. The direction in all these lines must be away from the common vertex.

(2) The three sides of any triangle may represent a resultant velocity and two components. Either direction in any side may represent the resultant. But then the direction

round the perimeter, of the sides representing components, must be contrary to that assumed for the resultant.

In either case, all the vectors involved must of course be drawn to the same scale. But the constructions remain valid, if all the lines make equal angles with the vectors that they represent, instead of being drawn parallel to them.

The broken line which represents the component vectors arranged consecutively in any order leads to the extremity of the line representing the resultant. The operation of tracing such a broken line is called **geometrical addition.** The elements represented and added being vector quantities, the result is known as a **vector sum.** And the fact that tracing the resultant is equivalent to tracing the broken line is conveyed in the form, The resultant is equal to the vector sum of its components. This is written in symbols, with a slight extension of meaning for the signs $(=)$ and $(+)$, $\widehat{v} = \widehat{v}_x + \widehat{v}_y + \widehat{v}_z$. An equality of this kind is called a **vector equation.** It is a useful succinct method of presenting the resultant as determining, and being determined by, the coexistent components.

9. When the reference-axes are rectangular, the analytical expressions become more convenient for calculating the speed and direction of v in terms of v_x, v_y, and v_z, or *vice versa.* If l, m, n, are the direction-cosines of v, we have

$$v_x = lv;\ v_y = mv;\ v_z = nv;\ v^2 = v_x^2 + v_y^2 + v_z^2. \tag{1}$$

If the path is parallel to one of the reference-planes, say XY; and α is the angle between v and X; the equations can be used in the simplified forms

$$v_x = v\cos\alpha;\ v_y = v\sin\alpha;\ tg\,\alpha = \frac{v_y}{v_x};\ v^2 = v_x^2 + v_y^2. \tag{2}$$

By the phrase "velocity parallel to a line," we shall always understand the **orthogonal projection** of resultant velocity upon

that line, which can never be greater than the velocity projected. The words "project" and "projection," when unqualified, are to be taken as applying to rectangular axes. X, Y, Z, will hereafter be drawn at right angles, unless the contrary is specified.

Let X_1 be any line, and δ the angle between it and v. Then

$$v_{x_1} = v \cos \delta. \tag{3}$$

But AB (Fig. 4), and any broken line joining its extremities, have the same projection upon any line whatever. Hence, if l_1, m_1, n_1, are the direction-cosines of X_1, we have

$$v_{x_1} = l_1 v_x + m_1 v_y + n_1 v_z. \tag{4}$$

If the coördinate x_1 is measured parallel to X_1, from the point (x_o, y_o, z_o), the general expression giving its value for (x, y, z) is

$$x_1 = l_1 (x - x_o) + m_1 (y - y_o) + n_1 (z - z_o). \tag{5}$$

On comparing the derivative of equation (5) with regard to time and (4), it will be seen that $v_{x_1} = \dfrac{dx_1}{dt}$, if the time-rates of x_o, y_o, z_o, l_1, m_1, n_1, are zero; but not in general otherwise. It is therefore not generally true that the time-rate of a coördinate length is the velocity parallel to that coördinate. But this is true, when the coördinate is measured from a point that is at rest, along a line that is not changing its direction.

We may then pass from relations among coördinates to those among velocities by a process of differentiation, if due precaution be used. The reverse process of integration develops coördinate relations out of velocity relations. For instance, from $v = \dfrac{ds}{dt}$ follows $s = \int v dt$, to be taken between limits fixed by the particular problem. Similarly $x = \int v_x dt$, etc. When the factor such as v or v_x is constant throughout the range of the integral, multiplication replaces integration.

The length traversed in time t_1 at the constant speed v_1 is $v_1 t_1$. The integration being a process of algebraic summation, it is concerned with the speed-element of velocity only.

It is a corollary of the discussion in this section that the resultant velocity and path are not affected by the particular reference-system chosen, if the choice be restricted to such coördinate-systems as satisfy the condition of having origins and axes relatively at rest. So that a given resultant velocity may be described in terms of an indefinite number of groups of (rectangular or oblique) projections, which are the time-rates of the corresponding coördinates under the condition supposed. Graphically these will be represented by the edges of the parallelopipeds which can be constructed on the same diagonal. If the axes are rectangular, any component of one group will be connected with all three components of another group by an equation of the type shown in (4) above.

10. There are several matters of convention with regard to signs which may be introduced appropriately at this stage, in preparation for subsequent mathematical work.

(1) Analytical proofs and developments will in general be carried through in the "standard" form, the coördinates and other variables being assumed positive, so far as this is permissible without unduly limiting the treatment. Terms will then appear with their actual signs; the introduction of negative values is usually possible without special discussion.

(2) The signs of such derivatives as $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, are fixed by considerations already mentioned (§ 5). If v in any operation of projection has its sign previously determined by the (arbitrary) choice of positive s, the rule that angles are to be measured between positive directions of lines must be adhered to, if there is to be a coherent system of signs. Let v be the

resultant of v_x and v_y (Fig. 5). Let the positive directions of X, Y, s, be so chosen that v_x and v_y are positive, while v is negative. Then the (actual) positive values of v_x and v_y are given by the equations

$$\left.\begin{aligned} v_x &= (-v)\cos(\pi+\alpha) = v\cos\alpha, \\ v_y &= (-v)\cos\left(\frac{\pi}{2}+\alpha\right) = v\sin\alpha. \end{aligned}\right\} \tag{1}$$

These results would not be affected by writing the angles $(\pi-\alpha)$, and $\left(\frac{3\pi}{2}-\alpha\right)$ respectively, to cover the other convention for positive angle.

Results concordant with these, however, may be obtained by considering the vector quantity to be essentially positive, and the angle to be measured between its forward direction and the positive direction of the line upon which projection is made. This is evident from the figure.

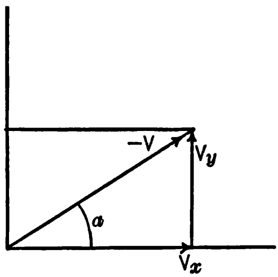


FIG. 5

(3) A nut, guided by a right-handed screw, is selected as the standard for positive turning about a line, in relation to positive direction along it. The angular displacement of the nut, and its advance along the axis of the screw, are considered to have the same sign. Hence on looking in the positive direction along a line, positive angular displacement about the line as an axis will be measured as the hands move on a clock dial that faces the observer, or "clockwise" (c.w.). For plane diagrams, the axis will ordinarily be assumed positive from the diagram toward the eye, and positive angle will then be seen as corresponding to counter-clockwise (c.c.w.) displacement.

(4) The (rectangular) axes X, Y, Z, will always be drawn so that turning through one quadrant positively about X brings Y to Z; about Y brings Z to X; about Z brings X to Y. This is known as the right-handed cyclical order for axes (see Fig. 2).

11. Let the path of Q lie in the plane of the diagram (Fig. 6), its position being given by means of the polar coördinates r and γ. Let the reference-system be O with OX, and v be the velocity of Q, making the angle β with r. We have two relations between the polar and the rectangular coördinates,

$$x^2 + y^2 = r^2; \quad (1)$$

$$\text{arc } tg \frac{y}{x} = \gamma. \quad (2)$$

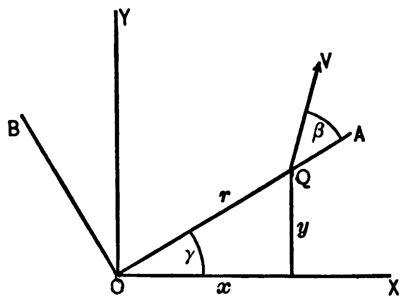


FIG. 6

Differentiate (1) with regard to time, divide by r, and write $\cos\gamma$ and $\sin\gamma$ for $\frac{x}{r}$ and $\frac{y}{r}$ respectively. We obtain

$$v_x \cos\gamma + v_y \sin\gamma = \frac{dr}{dt}. \quad (3)$$

The left-hand member is the velocity parallel to a line making an angle γ with OX; drawn from O, the line will coincide with r at the instant. This velocity is shown to be $\frac{dr}{dt}$ in magnitude, although the coördinate-line r is not in general at rest. The special conditions that determine this result will be examined later (§ 15). Consistently with the notation previously introduced, $\frac{dr}{dt}$ will be denoted by the symbol v_r.

It will be positive in the direction away from the pole, since the radius-vector is assumed positive outward. Note that

$$v_r = v \cos \beta.$$

Differentiate (2) with regard to time. We obtain

$$\frac{xv_y - yv_x}{r^2} = \frac{d\gamma}{dt};$$

and therefore

$$v_y \cos \gamma - v_x \sin \gamma = r\frac{d\gamma}{dt}. \tag{4}$$

The left-hand member is the velocity parallel to a line making an angle $\gamma + \frac{\pi}{2}$ with OX, which may be drawn from O, perpendicular to r at the instant. This velocity is shown to be $r\frac{d\gamma}{dt}$, when expressed in terms of polar variables. It will also be written v_γ. Note that it is equal to $v \sin \beta$; and is positive in the direction OB (Fig. 6).

The factor $\frac{d\gamma}{dt}$ is called **angular velocity** and will be denoted by the symbol ω. It measures the rate at which the radius-vector changes its direction in the plane. The sign of angular velocity is determined by the considerations of § 5, in connection with the convention for positive angle (§ 10); it is measured in radians per second.

Simple combination of (3) and (4) yields the equations

$$v^2 = v_r^2 + v_\gamma^2; \quad tg\,\beta = \frac{v_\gamma}{v_r}; \tag{5}$$

$$v_x = v_r \cos \gamma - v_\gamma \sin \gamma; \quad v_y = v_\gamma \cos \gamma + v_r \sin \gamma. \tag{6}$$

If we regard equations (6) as vector equations, and add them, the resulting equation is

$$\hat{v}_x + \hat{v}_y = [\hat{v}_r \cos \gamma + \hat{v}_r \sin \gamma] + [\hat{v}_\gamma \cos \gamma - \hat{v}_\gamma \sin \gamma] = \hat{v}_r + \hat{v}_\gamma. \tag{7}$$

These justify the graphic relation shown in Fig. 7, where v is drawn as the common resultant of (v_x, v_y) and (v_r, v_γ); and

the fact is exhibited that the operation of projection is reciprocal between v_x and v_y on the one hand, and v_r and v_γ on the other. Each one of either pair is the algebraic sum of the projections of both of the other pair upon its line (compare § 9).

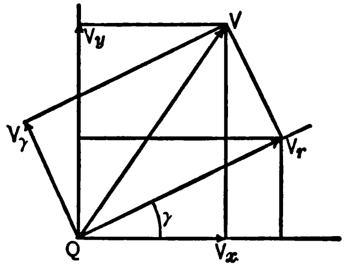


FIG. 7

The equations (3) and (4) above form the basis of the statement that in using polar (plane) coördinates, the actual motion (though describable otherwise) is considered as resulting from motion in a circle of radius r with the pole as centre, combined with change in length of the radius.

12. It is sometimes required to express a resultant velocity, or velocity parallel to any line, in terms connected with coördinates measured from points that move, and along lines that make variable angles with the reference-system. Instead of deducing the necessary relations in a general form, it will suffice to take up two special cases. They are selected as yielding the results which, when combined, cover the range of ordinary problems. The discussion of the first is begun in this section; the second is stated in § 15.

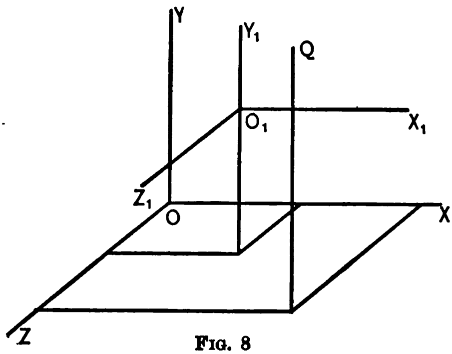


FIG. 8

Let the reference-system be O, with X, Y, Z (Fig. 8). Re-

ferred to these elements the coördinates of Q are x, y, z; and of O_1 are x_o, y_o, z_o. The coördinates of Q measured from O_1, always parallel to X, Y, Z, respectively, are x_1, y_1, z_1. The resultant velocity of Q is to be expressed in terms derived from (x_o, y_o, z_o) and (x_1, y_1, z_1). The following relations are evident:

$$x = x_o + x_1; \quad y = y_o + y_1; \quad z = z_o + z_1. \tag{1}$$

If Q and O_1 are both supposed to be moving,

$$\frac{dx}{dt} = \frac{dx_o}{dt} + \frac{dx_1}{dt}; \quad \frac{dy}{dt} = \frac{dy_o}{dt} + \frac{dy_1}{dt}; \quad \frac{dz}{dt} = \frac{dz_o}{dt} + \frac{dz_1}{dt}. \tag{2}$$

Equations (2) meet the requirement; for the projections of Q's velocity, and therefore the resultant, can be found in terms of the given type. But the same equations are open to another interpretation, which may easily be made if they are summarized in one vector equation. The left-hand members are projections of a velocity that may be written $v_Q(O)$. In this notation the subscript denotes the point whose motion is under consideration; and the letter within brackets the origin assumed for the reference-system. Similarly the first terms of the right-hand members are projections of $v_{O_1}(O)$; and the second terms are projections of $v_Q(O_1)$. For, if the reference-system were O_1 with X, Y, Z, the projections of Q's velocity would be $\frac{dz_1}{dt}, \frac{dy_1}{dt}, \frac{dz_1}{dt}$. The vector equation is, therefore,

$$\widehat{v}_Q(O) = \widehat{v}_{O_1}(O) + \widehat{v}_Q(O_1). \tag{3}$$

This form of result is independent of any particular choice of the point O_1, and the directions of the axes X, Y, Z, provided only that the coördinates x_1, y_1, z_1, are measured parallel to those axes. Equation (3) may be embodied in a verbal statement as follows: The velocity of a point can be expressed as the resultant of two elements. (1) The velocity of any second point. (2) The velocity of the first point if the

second were assumed as origin, the reference-axes being retained. The idea thus developed will be referred to as the principle of **change of origin for velocities.** It is seen to cover those cases in which coördinates of unchanging direction are measured from points that move. If O_1 is a fixed point (or even instantaneously at rest), $\bar{v}_{O_1}(O) = 0$, and $\bar{v}_Q(O) = \bar{v}_Q(O_1)$, as previously shown (§ 9).

13. Transposing one term in each of equations (2) of the preceding section, we may write,

$$\frac{dx_1}{dt} = -\frac{dx_o}{dt} + \frac{dx}{dt}; \quad \frac{dy_1}{dt} = -\frac{dy_o}{dt} + \frac{dy}{dt}; \quad \frac{dz_1}{dt} = -\frac{dz_o}{dt} + \frac{dz_1}{dt}. \tag{1}$$

Combining these into a vector equation, and noticing that the reversal of all components of a velocity reverses the velocity itself, the result is

$$\bar{v}_Q(O_1) = [-\bar{v}_{O_1}(O)] + \bar{v}_Q(O). \tag{2}$$

This form is especially adapted to the case where the data are velocities of two points for the same reference-system, and the relative velocity of the two points is sought, the reference-axes being retained. Equation (2) might be derived directly from (3) of the preceding section by transposition with change of sign. We may not only recognize and interpret such a possibility of transposition, on comparing the two forms of equation, but also formulate their connection in the following general rule: If v is a resultant vector with components v_1, and v_2, either component will be the resultant of v and the other component reversed. The parallelogram $ABCD$ of Fig. 9 exhibits this relation graphically.

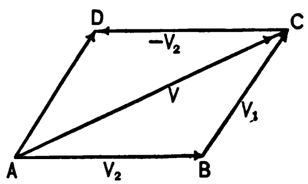


FIG. 9

Such a process of changing origin as is effected according to equation (2) may be described as the addition of equal components at all the points considered, the element thus geometrically added being so chosen as to neutralize the previous velocity of the new origin. The same truth is presented from another point of view, when it is said that equal component velocities may be added at two points, without affecting their velocities relative to each other.

14. The composition of oblique component velocities into a resultant (§ 7) can be treated according to the scheme of § 12. Using Q_1 (Fig. 3) as an "intermediate" origin, the equation may be written $\widehat{v}_Q(O) = \widehat{v}_x + \widehat{v}_y$, since $\widehat{v}_Q(Q_1)$ is v_y.

Further it is possible, instead of using one point O_1, to assume any number of points O_1, O_2, O_3, etc., in succession as intermediate origins, having any assigned velocities, before arriving at the final origin O. The velocity of Q referred to O may then be regarded as built up from a series of components, according to the equation

$$\widehat{v}_Q(O) = \widehat{v}_Q(O_1) + \widehat{v}_{O_1}(O_2) + \cdots + \widehat{v}_{O_n}(O). \qquad (1)$$

Graphically, the resultant velocity of Q will be represented by one side of a (plane or gauche) polygon; and the series of components forming the right-hand side of such an equation as (1) will be represented by the rest of the perimeter. An indefinite number of polygons can be drawn, starting with one given resultant. The principle of the triangle of velocities (§ 8) is here extended, and includes the idea of the **polygon of velocities,** which may be stated as follows: Any polygon, plane or gauche, may represent a resultant velocity and a group of its components. Either direction in any side may represent the resultant. But the components must then be taken so as to follow consecutively round the perimeter in the contrary direction.

15. Let Q move in the plane of the diagram (Fig. 10), the reference-system being O with X and Y. Let X_1, Y_1, be a pair of rectangular coördinate-axes in the same plane, and also intersecting at O. The axes X_1, Y_1, may move round O in the plane XY, so that X_1 makes a variable angle γ with X. It is required to express the velocities parallel to X and Y, and through them the velocity parallel to any line, and the resultant velocity, in terms of x_1, y_1, γ (and their time-rates), coördinates connected with lines that change their directions.

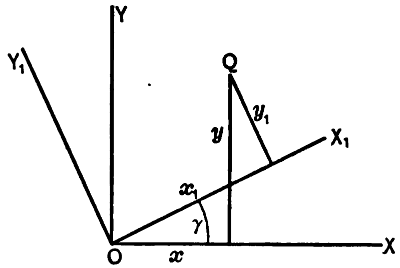


FIG. 10

The coördinate relations are

$$x = x_1 \cos\gamma - y_1 \sin\gamma;\ \ y = y_1 \cos\gamma + x_1 \sin\gamma. \tag{1}$$

Hence also:

$$\left.\begin{aligned} \frac{dx}{dt} &= \frac{dx_1}{dt}\cos\gamma - \frac{dy_1}{dt}\sin\gamma - x_1 \sin\gamma\frac{d\gamma}{dt} - y_1\cos\gamma\frac{d\gamma}{dt}, \\ \frac{dy}{dt} &= \frac{dy_1}{dt}\cos\gamma + \frac{dx_1}{dt}\sin\gamma - y_1 \sin\gamma\frac{d\gamma}{dt} + x_1\cos\gamma\frac{d\gamma}{dt}. \end{aligned}\right\} \tag{2}$$

Equations (2) meet the requirement imposed; but they may usefully be thrown into somewhat different form. First, substituting x and y from (1), and setting $\omega = \frac{d\gamma}{dt}$, we may write,

$$\left.\begin{aligned} \frac{dx}{dt} &= \frac{dx_1}{dt}\cos\gamma - \frac{dy_1}{dt}\sin\gamma - y\omega, \\ \frac{dy}{dt} &= \frac{dy_1}{dt}\cos\gamma + \frac{dx_1}{dt}\sin\gamma + x\omega. \end{aligned}\right\} \tag{3}$$

Secondly, equations (2) may be combined, and made explicit for $\frac{dx_1}{dt}$ and $\frac{dy_1}{dt}$. The results are

$$\left.\begin{aligned}\frac{dx_1}{dt} &= \frac{dx}{dt}\cos\gamma + \frac{dy}{dt}\sin\gamma + y_1\omega, \\ \frac{dy_1}{dt} &= \frac{dy}{dt}\cos\gamma - \frac{dx}{dt}\sin\gamma - x_1\omega.\end{aligned}\right\} \tag{4}$$

If, as a special case, X_1 and Y_1 move in such a manner that X_1 always contains Q, y_1 and $\frac{dy_1}{dt}$ become zero. Equations (4) then assume the simplified forms,

$$\left.\begin{aligned}\frac{dx_1}{dt} &= \frac{dx}{dt}\cos\gamma + \frac{dy}{dt}\sin\gamma, \\ x_1\omega &= \frac{dy}{dt}\cos\gamma - \frac{dx}{dt}\sin\gamma.\end{aligned}\right\} \tag{5}$$

Between these last expressions and equations (3) and (4), § 11, there is no difference save in notation. The results there deduced for polar coördinates are consequently shown to be syncopated forms of the more general relations developed above.

16. Equations (3) and (4) of the preceding section may be so discussed as to make evident what change is produced in the velocity of a point when new reference-axes are taken, under the limitations there announced, which include retention of the origin.

The left-hand members of (3) have for their resultant the velocity of Q referred to O with X, Y. Write this $v_Q\,(XY)$. The first two terms in the second members have a resultant $v_Q\,(X_1Y_1)$, the letters within brackets denoting as before the reference-axes. The last terms in the second members are projections of a velocity whose (numerical) magnitude is

$$\omega\sqrt{x^2 + y^2} = \omega r.$$

It makes with the X axis an angle whose tangent is $-\frac{x}{y}\cdot$ It is, therefore, perpendicular to r. And ω being considered positive in the standard case, the signs of the projections are seen to indicate a velocity corresponding to positive rotation; *i.e.* the rotation of X_1 relative to X. Hence equations (3), taken together, are equivalent to the vector equation

$$\widehat{v}_Q(XY) = \widehat{v}_Q(X_1Y_1) + \widehat{r\omega}. \qquad (1)$$

Analyzing equations (4) in similar fashion, we obtain

$$\widehat{v}_Q(X_1Y_1) = \widehat{v}_Q(XY) + (-\widehat{r\omega}). \qquad (2)$$

The plane (X_1Y_1) may be thought of as moving over the plane (XY), the origins remaining superposed. If the point Q is at rest in the plane (X_1Y_1), it will be moving with velocity $+\widehat{r\omega}$, perpendicular to r, in the plane (XY). If, on the other hand, Q is at rest in the plane (XY), it will be moving perpendicularly to r, with velocity $-\widehat{r\omega}$, in the plane (X_1Y_1). The velocity $\pm r\omega$ constitutes the element of difference between the two resultant velocities of Q. And, in general, the velocity in either plane is found by adding (geometrically) to the velocity in the other, the product of the radius-vector of Q by the angular velocity of the latter plane relatively to the former. Notice particularly that ω is the angular velocity, not of r but of X_1. If $\omega = 0$, $\widehat{v}_Q(XY) = \widehat{v}_Q(X_1Y_1)$ (§ 9).

The results of the present section may be combined with those of § 13, in treating any case where the changes of origin and reference-axes both become advisable, which have been considered separately in the text.

17. The bodies whose motions it is the task of Mechanics to describe occupy volumes within which there may be continuous or discontinuous distribution of material. In order to prepare the way for the discussions that become necessary later,

our kinematics must include the consideration of motions in groups or systems of points. These will not in general be independent of each other, but subject to constraints, given in the shape of equations of condition. The motions of the individual points proceed within the limitations imposed by the constraints, and expressed in the equations of condition. The treatment of the following chapters is in the main restricted to **rigid bodies**. A body is rigid, in this technical sense, if it suffers no change in volume, external shape, or internal configuration, under the action of forces applied to it. The preliminary kinematics should accordingly give prominence to the case of a rigid (geometrical) solid. The system of simultaneous velocities throughout such a solid is plainly governed by the condition that the distance between any two of its points does not change. It is found that all possible displacements of the solid, subject to the condition of invariable distance between any one point, and every other, can be described in terms of two types of motion. The actual system of velocities for points of the solid may conform to one type purely; or it may be the resultant at every point of components that would arise from the two types singly.

18. If all lines joining points of the solid retain their directions relative to the reference-system, the type of motion is known as **translation**. It is therefore characteristic of translation that all positions of the same line are parallel. Hence the simultaneous velocities of all points in the solid are equal, though they do not necessarily remain constant in either speed or direction. In order to show this, take A and B (Fig. 11) as the positions of any two points at $t = 0$; A' and B' being their positions at $t = \Delta t$. The curves AA' and BB' are their paths for the interval. Draw AB and $A'B'$. These lines will be equal in length, because the solid is supposed rigid; and they

will be parallel if the motion be one of translation. The latter condition is evidently consistent with curved paths for A and B. Hence the chords AA' and BB' are parallel and equal for any value of Δt. The simultaneous velocities of A and B are determined by the directions of the chords and the values of $\frac{AA'}{\Delta t}$ and $\frac{BB'}{\Delta t}$, all for the limit $\Delta t = 0$; they are, therefore, equal. And since B is chosen arbitrarily in connection with A, the conclusion holds for all points of the solid. The instantaneous velocity common to all points of the solid under these conditions is spoken of as the **velocity of the solid.**

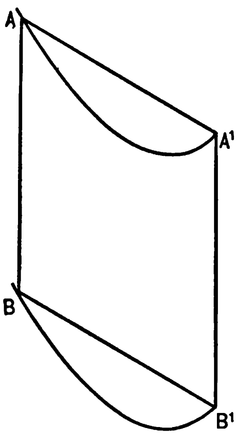


FIG. 11

It follows as a corollary of equality in the velocities that the paths of all points of a rigid solid which has a motion of translation will be like curves, tangents being parallel at simultaneous positions.

It is instructive to remark that the modification in the system of velocities, introduced by a change of origin, amounts to giving an additional motion of translation in the case of a rigid solid, since the same velocity is added at every point ($v_{o_1}(O)$ or $v_o(O_1)$, §§ 12, 13).

19. The second type of motion is **rotation.** The rigid solid is turning about an axis. All points of this line are at rest. The points of the solid are moving in circles with centres on the axis and planes perpendicular to it. The speed of any point in its circle can be expressed as $r\omega$, where r is the distance of the point from the axis, and ω is the angular velocity of r. From the fundamental condition for a rigid solid it fol-

lows that the simultaneous values of ω must be the same for all points. This common value is called the **angular velocity of the solid**; it may, of course, be constant or variable. The convention for sign of angular velocity has already been established (§ 10). Speed of rotation is frequently given as a number of revolutions per minute. If n is that number, and ω the angular velocity of the solid, $\omega = \dfrac{2\pi n}{60}$. If n is the number of revolutions per second, $\omega = 2\pi n$.

The angular velocity of the solid also measures the time-rate of change in direction for all lines of the solid which lie in planes perpendicular to the axis.

Such changes of reference-axes as are discussed in § 16 add an element of rotation about the Z axis in the case of a rigid solid.

20. Translation and rotation involve each a system of related velocities throughout the solid; and the actual group of velocities will not in general conform to either one of these systems. The truth of the general statement at the end of § 17 will not be demonstrated at this stage; but in one limited case of importance for present purposes it will be proved that the types of translation and rotation are sufficient when combined to include all the motions possible within the limitations.

The case to be considered is that in which a rigid solid moves in such a manner that the velocities of all its points remain parallel to a fixed plane, often called the guide-plane. It is evident that, under these conditions, simultaneous velocities of all points in the same normal to the guide-plane must be equal. Hence, the whole velocity-system may be represented unchanged in the guide-plane, or any fixed plane parallel to it; and, accordingly, the motion is known as **uniplanar**.

Let the diagram (Fig. 12) be drawn in the guide-plane. ABD is the projected outline of the solid; and P, Q, Q_1, are any three of its points, as projected upon that plane. The ref-

erence-system is O, with X, Y. If the origin should now be changed to P, the reference-axes being retained, the only possible motion of the solid for that origin would be a rotation about an axis through P, and normal to the guide-plane. For all the velocities must still remain parallel to the guide-plane, and all triangles drawn in the solid must move without change, either in their actual dimensions, or in their projections upon the guide-plane.

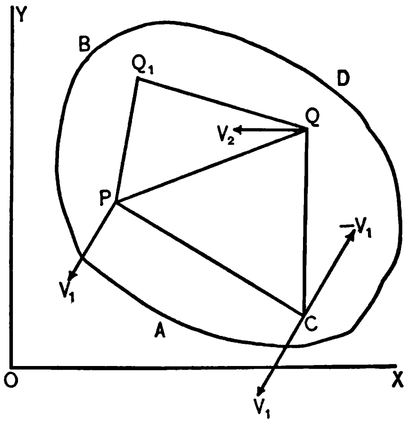


FIG. 12

The change of origin from P back to O is equivalent to adding a motion of translation equal to the velocity of P relative to O (§ 18, end). Since P is taken arbitrarily, the conclusion is valid that the uniplanar motion of a rigid solid can in general be described as a rotation about an axis drawn through any one of its points normal to the guide-plane, combined with a translation parallel to that plane. Zero values of these elements are, of course, possible as particular cases.

The same line of reasoning would hold if Q or Q_1 had been chosen as origin, instead of P. Hence, the actual motion relative to O and X, Y, can be described in terms of an indefinite number of similar combinations. The angular velocity of any such triangle as PQQ_1 is the same under the given conditions, whichever vertex the axis is assumed to pass through; and, therefore, the element of rotation is unaffected in magnitude and sign by the location of the axis at a par-

ticular point in the solid. The element of translation, however, does depend upon the point selected, being always equal to the velocity of that point relative to O. The same analysis of the motion holds true, even though P is not taken within the boundary of the solid, as ordinarily understood. The point through which the axis of rotation is conceived to pass may be any outlying point, provided only that the latter is rigidly attached to the solid, and participates in its motion.

21. It may happen that the points of the solid in one normal to the guide-plane are at rest relatively to O, either permanently or instantaneously. If the former, the solid is evidently in pure rotation about that normal, and the system of velocities becomes simplified in expression by taking the origin where the axis cuts the guide-plane. The case of instantaneous rest is of such particular interest and practical importance that it calls for somewhat independent discussion.

Let points of a normal drawn in the solid and represented at P (Fig. 12) have zero velocity at the instant considered. Then the velocities referred to P and to O as origins will be instantaneously the same for all points of the solid (§§ 9, 12), and the system of velocities will be like that for pure rotation, with the normal to the guide-plane at P as axis. When this condition is fulfilled for any normal, that line of the solid is called the **instantaneous axis**; and the point at which it cuts the guide-plane is the **instantaneous centre**. Or, with a slightly different aspect to the thought, the same terms are applied to the fixed point and line with which P and the normal through it coincide at the instant.

As the motion proceeds, different lines in the solid become in turn the instantaneous axis, each at its own epoch. This group of lines may be determined in advance as a cylindrical locus moving with the solid. The group of fixed lines, indi-

cating the position which each instantaneous axis will occupy as its turn arrives, also constitutes a cylindrical locus, fixed relatively to the reference-system. As the motion goes on, the surface of the moving cylinder is rolled or developed on that of the fixed cylinder, the elements being brought to coincidence in pairs. The sections of the cylinders perpendicular to their axes appear in the guide-plane as curves, whose character depends upon the motion, and which are known as **fixed** and **moving centrodes**, respectively. In problems of machine-design the conception of uniplanar motion as resulting from the rolling of the moving upon the fixed centrode is especially serviceable.

22. The instantaneous axis for the solid is convenient, because the velocity of every point can be directly expressed in terms of its distance r from that axis, and the angular velocity ω. The velocity of any point is $r\omega$, and is perpendicular to r. Therefore constructions and calculations based on this idea are frequently used. The details of locating the instantaneous centre depend upon the form in which the data appear. One case that often presents itself is that in which the velocities in the guide-plane of two points are given. Let P and Q (Fig. 12) represent in the guide-plane any two points of the solid, their velocities being v_1 and v_2, and the reference-system O, with X, Y. Draw through P and Q perpendiculars to v_1 and v_2. If these lines intersect at C, that point is the instantaneous centre. For, if we draw PQ, it may represent the velocity of Q for the reference-system P with X, Y, which is $\omega \cdot \overline{PQ}$, making an angle $+\frac{\pi}{2}$ with PQ if ω is positive. Then CPQ being taken as a velocity triangle, CP will properly represent v_1 as $\omega \cdot \overline{CP}$ at the same angle $+\frac{\pi}{2}$ with CP; and CQ will represent v_2 as $\omega \cdot \overline{CQ}$ at $+\frac{\pi}{2}$ with CQ; if C be the instan-

taneous centre. Moreover, the necessary relation is satisfied: $\widehat{v}_2 = \widehat{v}_1 + \widehat{\omega \cdot \overline{PQ}}$ (§§ 18, 20), because in the triangle CQ represents a resultant and CP, PQ, its two components.

If the directions only of the two velocities v_1 and v_2 are given, the instantaneous centre may still be located, and the velocities of all points become determinate in direction, for an assumed sign of ω. If in addition the magnitude of one velocity is known, ω becomes fully known, and the magnitude of any velocity can be calculated.

If v_1 and the angular velocity round the axis through P are given, the latter quantity being necessarily equal to the angular velocity round the instantaneous axis (§ 20, end), C may be viewed as the point for which the translation element v_1 and the rotation element $\omega \cdot \overline{PC}$ are equal and opposite, giving zero resultant (see Fig. 12).

If the known velocities v_1 and v_2 are in parallel lines, the construction given above fails. If they are equal, the motion must be one of translation, and there is no instantaneous centre at a finite distance. The centre can be located easily in the common normal to the two velocities, if they are equal in magnitude and contrary in direction; or unequal in magnitude, and have the same or contrary directions in parallels.

In any case of rotation, the axis may lie outside of the solid that revolves about it. So the point C may not fall within the boundary ABD (Fig. 12). But a plane of any required extent, coinciding with the guide-plane, may be conceived as attached to the solid and moving with it, for the purposes of the construction. Or the instantaneous axis may be looked at from the other point of view (§ 21), as a fixed line so chosen as to give simplest expression to the system of velocities. The discussion can be accommodated to either aspect of the conception.

CHAPTER II

ACCELERATION

23. The velocity of a point is variable when either the speed or the direction of the velocity is changing. The projection of variable velocity upon any fixed line will in general also be variable, and therefore the projection of the moving point upon such a line will move with varying speed. There is then said to be **acceleration** parallel to the line; and we are at liberty to associate the acceleration either with the velocity of the projection, or with the projected velocity of the moving point, since these are equivalent descriptions of the same quantity (§ 7). It can be seen at once that accelerations of points must be dependent upon the reference-system employed; for velocity cannot be specified as constant or variable until these elements have first been selected.

Let a fixed line be taken, and called X; so that v_x is the quantity with which acceleration is to be connected. When v_x is variable, it is permissible to regard it as modified by a process of algebraic addition; and the conception of acceleration is so formulated as to measure the intensity of such a process. **Acceleration parallel to X** is defined as the time-rate of v_x; or $p_x = \frac{dv_x}{dt}$, the letter p being adopted to denote acceleration, and the subscript indicating direction, as in the case of velocity. That is, the algebraic addition to v_x is being carried on at a rate which would produce a change of amount p_x in one second if maintained. Consequently the acceleration in question may itself be classed as a vector quantity, with

direction parallel to X, and magnitude p_x. It may therefore be represented by a line drawn in its direction, whose length shows the magnitude to any chosen scale. The sign of the coefficient $\frac{dv_x}{dt}$ follows the rule already repeatedly referred to; so that the vector p_x will have the positive or the negative direction of X, according as v_x is increasing or decreasing algebraically. As a technical term, then, acceleration includes the notion of retardation also. But it should not be overlooked that, if the velocity is in the negative direction ($v_x < 0$), *positive* value of p_x corresponds to a motion that is becoming *slower; i.e.* to a retarded motion in the popular sense of the word.

The coefficient $\frac{dv_x}{dt}$, which is the fundamental value of p_x, may be expressed in other forms, since $v_x = \frac{dx}{dt}$. Thus we may write,

$$p_x = \frac{d^2x}{dt^2} = \frac{v_x\,dv_x}{dx} = \frac{d}{dx}\left(\frac{v_x^2}{2}\right). \tag{1}$$

The unit naturally suggested for the numerical measure of acceleration is a change in one second amounting to one unit of speed. Consistently with the standards previously introduced (§ § 3, 5), the unit of acceleration is [one centimeter per second] per second $\left(1\ \frac{\text{cm.}}{\text{sec.}^2}\right)$. Because the elements of direction and magnitude are both essential, however, accelerations will not be accepted as equal, unless they have the same direction along parallel lines, and their magnitudes are equal also.

24. Let a point move in a plane curve with velocity v, and X be any fixed line in that plane. If α is the angle made by v with X, $v_x = v \cos \alpha$, and consequently,

$$p_x = \frac{dv_x}{dt} = \frac{dv}{dt}\cos\alpha - v\frac{d\alpha}{dt}\sin\alpha. \tag{1}$$

Suppose X so chosen that $\alpha = 0$ at the instant. Then, as a particular value, $p_x = \frac{dv}{dt}$. This special choice of the line X can always be made. If that be done at each point of the path, the resulting quantity $\frac{dv}{dt}$ will be continually an element of acceleration in the line with which the tangent for the instant coincides. It is therefore called **tangential acceleration,** and the symbol appropriated to it is p_t. As it is a particular case of the general value p_x, the considerations connected with sign need not be repeated. The alternative forms are at once seen to be

$$p_t = \frac{dv}{dt} = \frac{d^2s}{dt^2} = \frac{vdv}{ds} = \frac{d}{ds}\left(\frac{v^2}{2}\right). \tag{2}$$

25. The fixed line X may be otherwise chosen, so that the angle α is $+\frac{\pi}{2}$ at each point of the path. Then the tangent and the line X are instantaneously at right angles, so that $v_x = 0$. But the particular value of the acceleration is $p_x = -v\frac{d\alpha}{dt}$. This quantity will be continually an element of acceleration in the line with which the normal for the instant coincides. It is therefore called **normal acceleration,** and will be denoted by the symbol p_n. The normal acceleration will have the direction of positive X, under the conditions assumed, if v and $\frac{d\alpha}{dt}$ have opposite signs; and of negative X, if those factors have the same sign. Hence the vector p_n points toward the concave side of a curved path at every position,

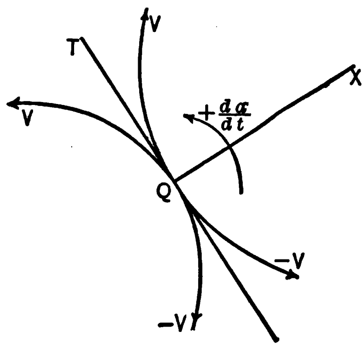


FIG. 13

as can be seen from Fig. 13, in which QX and QT are the positive directions of X and the tangent.

Curvature being defined as the change in direction of the tangent per unit length of arc, and radius of curvature (ρ) as the reciprocal of curvature, we have $\rho = \frac{ds}{d\alpha}$, and $\frac{d\alpha}{dt} = \frac{v}{\rho}$. Combining this with the original expression for normal acceleration, the alternative form is obtained, $p_n = -\frac{v^2}{\rho}$.

26. If the speed is not changing, $\frac{dv}{dt} = 0$, and the tangential acceleration vanishes, although the path be a curve. Otherwise expressed, the function of this element in acceleration is to bring about change in speed, or rapidity of motion, only.

If the direction of the velocity is not changing, $\frac{d\alpha}{dt} = 0$, and the normal acceleration vanishes, although the speed be variable. That is, the function of acceleration which stands in this relation to the path is to produce in velocity a change of direction only. It should be noticed, also, that normal acceleration assumes the value zero whenever the speed does so.

Fixed lines chosen at each position of a point moving in a plane curve, so as to coincide with the tangent and the normal to the path at that instant, are thus seen to possess the advantage of exhibiting the two functions of acceleration separated. By virtue of this property such lines are convenient rectangular axes, because they yield simplified special values for the accelerations parallel to them; although the advantage connected with each pair of these lines is transient, and confined to a single point of the path in the general case. The accelerations parallel to other lines, chosen at right angles but arbitrarily, contain two terms combined; one due to change in speed, and another to change in direction. For it is evident from the very form of equation (1), § 24, that any such vector

as p_x is the algebraic sum of the projections of p_t and p_n upon its line.

27. Let Q be a point moving in the plane of the diagram (Fig. 14). Draw QA and QC representing the tangential and normal accelerations, and QX showing the direction of any fixed line X. Complete the rectangle $QCBA$, and project B upon QX at D. Then the sum of the projections of QA and QC upon X will be equal to QD, which will therefore represent p_x. But QD is likewise the projection of QB on X; and in general the acceleration parallel to any line in the plane will be represented by the projection upon that line of QB, which is the diagonal drawn from Q, of the rectangle described upon QA and QC. Hence the acceleration parallel to that diagonal is a maximum value. It is equivalent to the two coexisting vectors p_t and p_n, and is called **resultant acceleration,** the tangential and normal accelerations being rectangular components of it. The symbol p will be assigned to this resultant vector; in magnitude $p = \sqrt{p_t^2 + p_n^2}$, and its direction makes an angle β with the tangent such that $tg\,\beta = \frac{p_n}{p_t}$.

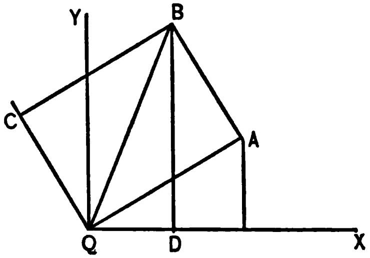


FIG. 14

The conception of resultant acceleration has been arrived at by means of a particular pair of components, p_n and p_t; but it is evident from the figure that the same quantity may equally well be regarded as the resultant of other pairs at right angles. For example, those represented by QD and DB, parallel to X and Y respectively, which are any fixed lines perpendicular to each other in the plane. The magnitude of p can then be expressed as $\sqrt{p_x^2 + p_y^2}$; and its direction relatively to X

by the angle δ, whose tangent is $\frac{p_y}{p_x}$. With reference-systems *fixed* relatively to each other, it is true *always* that the same resultant velocity of a given moving point is determined by any set of components. But then the resultant acceleration must be the same for all sets also. It is therefore a direct consequence of the reasoning in § 9, that the resultants derived from any such pairs of terms as p_x and p_y should be identical; and that the projection of both p_x and p_y upon any line must give the acceleration parallel to that line (cf. § 34).

28. The conclusions that have been drawn by a method based upon graphical relations in the diagram can of course be verified analytically as well. Reproducing equation (1), § 24, and adding a similar expression for acceleration parallel to Y,

$$\left.\begin{aligned} p_x &= p_t \cos\alpha - p_n \sin\alpha, \\ p_y &= p_t \sin\alpha + p_n \cos\alpha. \end{aligned}\right\} \tag{1}$$

These equations can be combined and made explicit for p_n and p_t:

$$\left.\begin{aligned} p_t &= p_x \cos\alpha + p_y \sin\alpha, \\ p_n &= p_y \cos\alpha - p_x \sin\alpha. \end{aligned}\right\} \tag{2}$$

Hence the projection-relations are reciprocal between (p_t, p_n) and (p_x, p_y); each one of either pair is obtainable by projecting upon its line both of the other pair. Further, if equations (1) be treated as vector equations, we find by addition

$$\widehat{p}_x + \widehat{p}_y = \widehat{p}_t + \widehat{p}_n. \tag{3}$$

This identity of the resultants can be established in like manner from (2). Calling the common resultant p, and assuming $tg\,\delta = \frac{p_y}{p_x}$; $tg\,\beta = \frac{p_n}{p_t}$, we have

$$p_x = p\cos\delta;\ p_y = p\sin\delta;\ p_t = p\cos\beta;\ p_n = p\sin\beta. \tag{4}$$

There is on the whole not so great a convenience in the direct expression of resultant acceleration, as has been found in using resultant velocity; the reason being that in general no geometrical element of the path is associated with the direction of the former, as the tangent is with that of the latter. Resultant acceleration at any position will fall in the tangent, or the line of resultant velocity, only when the normal component is zero. This condition is not permanently satisfied unless the path is a straight line. Resultant acceleration will lie permanently in the normal only when the speed is constant. In other cases where acceleration exists (apart from instantaneous exceptions, § 26), its direction will fall in one of the angles between the tangent and the "inward" normal. Of course $p = 0$ means $p_t = p_n = p_x = p_y = 0$.

29. The idea of acceleration parallel to a given line, whose definition was at first treated as fundamental, is now seen to be only preliminary. It is absorbed into the fuller conception of resultant acceleration, which in itself measures the effect produced by two concurrent processes of algebraic addition to existing velocity vectors. Or, if resultant acceleration be made more distinctly the primary thought, we may say it presents the total quantity which becomes apparent in detail through the partial changes affecting velocity in magnitude or direction. The way has thus been prepared for the consideration of resultant acceleration more directly, as involving a single process of geometrical addition, which replaces the two simultaneous processes of algebraic addition (§ 8). A similar equivalence between geometrical and algebraic addition has been brought out under the heading Velocity.

Let AC (Fig. 15) represent the velocity v at the time t, of a point moving in a plane curve; AB its velocity $v + \Delta v$ at the time $t + \Delta t$, the direction having changed by the angle $\Delta\alpha$.

Then AB is the geometrical sum of AC and CB; and the latter line represents the velocity component required to change the vector v into the vector $v + \Delta v$. Call that component Z. The components $2\,v \sin\frac{\Delta\alpha}{2}$ and Δv, represented by CD and DB, may together be substituted for Z. The vector Z will change in both magnitude and direction with the length of the time-interval Δt. We proceed to show that the resultant acceleration at the time t has for direction the direction of Z, and for magnitude the coefficient $\frac{Z}{\Delta t}$, both for the limit $\Delta t = 0$.

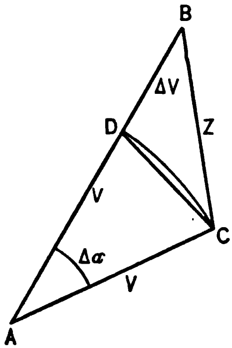


FIG. 15

At the limit, the triangle BCD is right-angled at D. Consequently,

$$\lim_{\Delta t = 0}\left(\frac{Z}{\Delta t}\right)^2 = \lim_{\Delta t = 0}\left(\frac{\Delta v}{\Delta t}\right)^2 + \lim_{\Delta t = 0}\left[2\,v\,\frac{\sin\frac{\Delta\alpha}{2}}{\Delta t}\right]^2$$

$$= \left(\frac{dv}{dt}\right)^2 + \left(v\,\frac{d\alpha}{dt}\right)^2 = p_t^2 + p_n^2. \qquad (1)$$

Also

$$\lim_{\Delta t = 0}\left[\frac{2\,v \sin\frac{\Delta\alpha}{2}}{\Delta v}\right] = \frac{v\,\frac{d\alpha}{dt}}{\frac{dv}{dt}} = \frac{p_n}{p_t} = tg\left[\lim_{\Delta t = 0}(DBC)\right]. \qquad (2)$$

Equations (1) and (2) establish the relations as announced between Z and the resultant acceleration. It can be seen from the diagram that CD, which at the limit takes the direction of the normal acceleration, must always be turned toward the concave side of the path.

30. Rectangular component accelerations parallel to fixed lines can be expressed in terms of polar coördinates. As in

the corresponding treatment of velocity, and in the use of tangential and normal accelerations, the fixed lines are to be differently chosen at each epoch. Such changes of axes are legitimate if they carry us through a series of elements which were equally open to choice at the outset, without affecting velocity, path, and acceleration. The later choice has the same range as that first exercised. Fixed lines satisfy this criterion (§§ 9, 27); and for the present purpose they must always be taken so that the radius-vector and a line making an angle $+\frac{\pi}{2}$ with it at the origin, coincide with them instantaneously. The accelerations parallel to such lines will be denoted by p_r and p_γ respectively, following the notation for the corresponding velocities (§ 11).

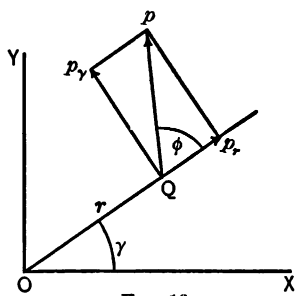


FIG. 16

Let Q move in the plane of the diagram (Fig. 16), O being pole, X and Y reference-axes, and γ polar angle. Then we may write at once (§§ 27, 28)

$$\left.\begin{aligned} p_r &= p_x \cos\gamma + p_y \sin\gamma, \\ p_\gamma &= p_y \cos\gamma - p_x \sin\gamma. \end{aligned}\right\} \quad (1)$$

All that remains is to express the right-hand members in terms of r, γ, and their time-derivatives. The coördinate relations that form the starting-point are

$$x = r\cos\gamma;\ y = r\sin\gamma. \quad (2)$$

A direct process of differentiating equations (2) with regard to time, and substituting in (1) yields

$$p_r = \frac{d^2 r}{dt^2} - r\left(\frac{d\mathrm{v}}{dt}\right)^2;\ p_\gamma = 2\frac{dr}{dt}\cdot\frac{d\gamma}{dt} + r\frac{d^2\gamma}{dt^2}. \quad (3)$$

The equation $\widehat{p}_r + \widehat{p}_\gamma = \widehat{p}_x + \widehat{p}_y,$ (4)

obtained by geometrical addition of the vector quantities in equations (1), emphasizes the thought that both pairs of components have a common resultant. We may write

$$p = \sqrt{p_r^2 + p_\gamma^2}; \text{ and } tg\,\phi = \frac{p_\gamma}{p_r};$$

where ϕ is the angle between p and the radius-vector.

The following alternative forms and items of notation are used in connection with equations (3):

$$\frac{dr}{dt} = v_r; \quad \frac{d\gamma}{dt} = \omega; \quad \frac{d^2r}{dt^2} = \frac{dv_r}{dt} = \frac{v_r dv_r}{dr} = \frac{d}{dr}\left(\frac{v_r^2}{2}\right);$$

$$\frac{d^2\gamma}{dt^2} = \frac{d\omega}{dt} = \frac{\omega\, d\omega}{d\gamma} = \frac{d}{d\gamma}\left(\frac{\omega^2}{2}\right).$$

As standard forms we shall employ

$$p_r = \frac{dv_r}{dt} - r\omega^2; \quad p_\gamma = 2\,\omega v_r + r\frac{d\omega}{dt} = \frac{1}{r}\frac{d}{dt}\,(r^2\omega). \qquad (5)$$

The positive direction of r is by convention taken outward along the radius-vector; consequently this must be the positive direction of p_r also. The positive direction of p_γ makes an angle $+\frac{\pi}{2}$ with r. The signs of these vectors as determined from equations (3) or (5) will depend upon the signs of the constituent terms. The coefficients $\frac{dv_r}{dt}$ and $\frac{d\omega}{dt}$ are positive or negative, according as the quantities v_r and ω are algebraically increasing or decreasing; $r\frac{d\omega}{dt}$ has the same sign as its second factor. The product $r\omega^2$ is always positive, and therefore the second element in the expression for p_r is always directed inward along r. The product $2\,\omega v_r$ contributes a positive term to p_γ, if ω and v_r have the same sign; a negative term, if these factors have contrary signs.

Since $\frac{d\omega}{dt}$ is the time-rate of angular velocity, this coefficient

is known as **angular acceleration.** It is measured numerically as [radians per second] per second $\left(\frac{\text{radians}}{\text{sec.}^2}\right)$.

The results of this section may profitably be compared with the parallel expressions for velocity (§ 11). First, it is not legitimate to think of the actual acceleration as the aggregate of those elements brought into play when motion in a circle and along a radius of the circle exist, each separately; although the resultant velocity may be so considered. The circular motion alone would in general require one acceleration $r\omega^2$ inward along the radius, and another $r\frac{d\omega}{dt}$ perpendicular to it. Motion along a radius alone (without rotation) would be covered by $\frac{dv_r}{dt}$. The term $2\,\omega v_r$ appears, in addition to the three former elements, in the actual result. As its form shows, it is dependent upon the coexistence of a rotation (ω), and a motion along the radius (v_r). Secondly, the acceleration parallel to the instantaneous direction of r is not $\frac{d^2r}{dt^2}$, as it would be if r were a fixed line, but $\frac{d^2r}{dt^2} - r\omega^2$, although the corresponding velocity is $\frac{dr}{dt}$ under either supposition. Such divergencies of type between the expressions for velocity and those for acceleration can be explained mathematically as due to the orders of the derivatives involved. A directer kinematical analysis of the values of p_r and p_γ is given in the next section, for the purpose of exhibiting each term as calculated from its function in changing the magnitude or the direction of a velocity.

31. At every instant, a point moving under the conditions laid down in § 30 has in general one component velocity v_r, and another v_γ. Since these are two components taken always in the same relations of direction to the radius-vector at any epoch, and the expressions for their magnitude are perma-

nently valid, they may be thought of as a pair of rectangular velocity-components, whose direction and magnitude are systematically changed in order to maintain a proper adjustment. Their directions must change at a rate which is the angular velocity of the radius-vector; and their magnitudes must always be the projections of the actual resultant velocity upon the instantaneous radius-vector and its perpendicular.

The acceleration required to change the speed of any velocity lies in the line of that velocity, and has for magnitude the time-rate of the speed. Change in direction is produced by an acceleration perpendicular to the velocity, having for magnitude the product of the speed by the rate of change in direction. Hence, the four elements of acceleration requisite in order to produce the necessary changes in direction and magnitude for both v_r and v_γ will be as follows:

$$\left.\begin{array}{l}\text{(1) To change the magnitude of } v_r;\ \dfrac{dv_r}{dt}.\\[2ex] \text{(2) To change the direction of } v_\gamma;\ v_\gamma\omega = r\omega^2.\end{array}\right\}\ \text{In the line of } r.$$

$$\left.\begin{array}{l}\text{(3) To change the magnitude of } v_\gamma;\ \dfrac{dv_\gamma}{dt} = r\,\dfrac{d\omega}{dt} + \omega v_r.\\[2ex] \text{(4) To change the direction of } v_r;\ v_r\omega.\end{array}\right\}\ \begin{array}{c}\text{Perpen-}\\ \text{dicular}\\ \text{to } r.\end{array}$$

The elements lying in the same line may now be assembled, their directions and signs being determined according to the rules already stated. We find in agreement with equations (5), § 30,

$$p_r = \frac{dv_r}{dt} - r\omega^2;\ p_\gamma = 2\,\omega v_r + r\frac{d\omega}{dt}.$$

The term $2\,\omega v_r$ is here shown to arise from the algebraic coalescence of an acceleration $v_r\omega$, required to change direction for v_r, and ωv_r, which is part of the acceleration producing change in the magnitude of v_γ.

32. In considering the elements which may enter into resultant acceleration, only orthogonal projection and rectangular components have thus far been spoken of. But the resultant can also be expressed in terms of accelerations belonging to oblique projections of the moving point.

Let Q move in the plane of the diagram (Fig. 17), X and Y', making an angle i, being drawn parallel to the reference-axes. Let the oblique projections of Q be Q_1 and Q_2. If their velocities are v_1 and v_2, their accelerations are $\frac{dv_1}{dt}$ and $\frac{dv_2}{dt}$, parallel to X and Y' respectively. The quantities v_1 and v_2 are in permanent adjustment to the resultant velocity; they vary in such a way as to remain always its oblique projections.

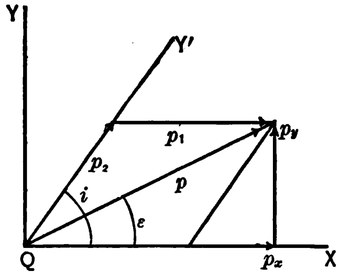


FIG. 17

The accelerations $p_1 = \frac{dv_1}{dt}$, and $p_2 = \frac{dv_2}{dt}$, when systematically applied to the velocities v_1 and v_2, are together equivalent to the resultant acceleration p, applied to the velocity v; that is, they are oblique components of p.

It can be shown without difficulty that the necessary algebraic conditions are fulfilled. Draw Y perpendicular to X (Fig. 17). Then in order that p may be the common resultant of (p_1, p_2) and (p_x, p_y), we must have,

$$\left.\begin{aligned} p_x &= p_1 + p_2 \cos i = \frac{dv_1}{dt} + \frac{dv_2}{dt}\cos i, \\ p_y &= p_2 \sin i = \frac{dv_2}{dt} \sin i. \end{aligned}\right\} \qquad (1)$$

But $v_x = v_1 + v_2 \cos i$; $v_y = v_2 \sin i$; and the time-rates of these expressions satisfy equations (1). In terms of p_1 and p_2, the

magnitude and direction of p are specified in the following relations, where the angle ϵ measures the inclination of p to X:

$$p^2 = p_1^2 + p_2^2 + 2p_1p_2 \cos i; \ \sin \epsilon = \frac{p_2}{p} \sin i. \qquad (2)$$

The phrase, "acceleration parallel to any line," will be limited as before to the result of orthogonal projection upon the line, which can never be greater than the quantity projected. A *component* acceleration (or velocity), which includes the possibility of oblique projection, may stand in any ratio of magnitude to the resultant.

Where the components may be oblique, the arguments and constructions of § 8 can be paralleled throughout. As applying to accelerations, the main results are restated below.

(1) Adjacent sides of a parallelogram, and the diagonal drawn through their intersection, may represent two component accelerations and their resultant, respectively. The directions in all these lines must be away from the common vertex.

(2) The three sides of any triangle may represent a resultant acceleration and two components. Either direction in any side may represent the resultant. But then the direction round the perimeter of the sides representing components must be contrary to that assumed for the resultant.

33. The discussions of §§ 23–32 have been confined to motion in a plane, chiefly in order to enforce the conception of acceleration as connected with change in direction and magnitude of velocity, without presupposing a knowledge of the more complicated geometry of space curves. So long, however, as component accelerations are taken parallel to the same axes, rectangular or oblique, at all positions of the moving point, the results already obtained can be extended to

three dimensions by inspection. And indeed it may become necessary to do this, although the path under consideration be a plane curve, when the axes are not assumed in that plane. If the axes X, Y, Z, constitute a rectangular reference-system, the acceleration parallel to Z can be written down at once:

$$p_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2} = \frac{v_z dv_z}{dz} = \frac{d}{dz}\left(\frac{v_z^2}{2}\right). \tag{1}$$

To determine the magnitude of the resultant acceleration, and its direction-cosines, we have

$$p^2 = p_x^2 + p_y^2 + p_z^2;\ l = \frac{p_x}{p};\ \ m = \frac{p_y}{p};\ \ n = \frac{p_z}{p}. \tag{2}$$

If the axes are oblique, the construction by means of an (oblique) parallelopiped (cf. § 8) can be applied to accelerations; and the requisite extension of equations (2), § 32, is evident, on comparing with § 7, equations (2).

Both the preceding forms of relation still find application instantaneously, if the axes are differently located at different instants. The element lacking in that treatment is a *systematic* connection or sequence among the successive choices, which would render expressions for magnitude of acceleration possible, in forms that remain valid when the axes are changed, and link together the separate values. For example, such as are derived when (fixed) axes in a plane are continually selected anew, so that at every instant the tangent and normal to the path, or the radius-vector and its perpendicular, are coincident with them.

34. It has been pointed out already (§ 23) that the acceleration of a point will in general be affected by shifting the reference-system. The effect of acceleration under given conditions being to carry velocity through a determinate series of changes in speed and direction, a new succession of velocities,

corresponding to another reference-system, will imply modified values for the acceleration. Consequently it becomes necessary to express the particular elements of difference imposed upon acceleration by a change of origin and reference-directions. In doing this, the same general plan will be pursued that guided the similar discussion for velocity (§§ 12, 16); and accordingly the **change of origin for accelerations** will be treated first, the directions of the reference-axes being retained.

If equations (2), § 12, be differentiated with respect to time, the results are

$$\frac{d^2x}{dt^2}=\frac{d^2x_0}{dt^2}+\frac{d^2x_1}{dt^2};\quad \frac{d^2y}{dt^2}=\frac{d^2y_0}{dt^2}+\frac{d^2y_1}{dt^2};\quad \frac{d^2z}{dt^2}=\frac{d^2z_0}{dt^2}+\frac{d^2z_1}{dt^2}. \qquad (1)$$

Regarding these as vector equations, adding them, and using here a notation like that adopted for velocity, we may write

$$\widehat{p}_Q(O)=\widehat{p}_{O_1}(O)+\widehat{p}_Q(O_1). \qquad (2)$$

If the acceleration of O_1 referred to O is zero, the accelerations of Q referred to O and to O_1 become identical. That is, the acceleration of a moving point is the same (the reference-directions being preserved) for all origins that have *zero acceleration* relatively to each other. The provisional limitation maintained in the earlier sections of this chapter, that accelerations, in order to be directly comparable, should be reckoned with reference to origins *fixed* relatively to each other, may now be removed. In the important case where the paths of two points relative to one another are straight lines described with constant speed, their relative acceleration is permanently zero, and either may be selected as origin without affecting the accelerations to be ascribed to other points.

Equation (2) may be written and used in the form

$$\widehat{p}_Q(O_1)=-[\widehat{p}_{O_1}(O)]+\widehat{p}_Q(O). \qquad (3)$$

Since $-[\widehat{p}_{O_1}(O)]$ is equal to $\widehat{p}_O(O_1)$, because $-[v_{O_1}(O)]$ is always equal to $\widehat{v}_O(O_1)$, equations (2) and (3) differ only as to the way in which the data are presented. The type of these equations is independent of any particular choice of O or O_1, and the reference-axes; provided always that the latter are assumed in the same directions at both origins. Either equation shows plainly the element of acceleration to be added geometrically, for all moving points involved, in transferring the reference from one origin (the first) to another (the second). It is the acceleration of the first origin referred to the second.

The final reference may be conceived as being reached after any number of steps, involving reference to intermediate origins, that move with accelerations assigned arbitrarily. We are, therefore, led to conclude that an acceleration may be regarded as composed of any number of partial accelerations. The vector equation to be satisfied is (cf. § 14)

$$\widehat{p}_Q(O) = \widehat{p}_Q(O_1) + \widehat{p}_{O_1}(O_2) + \widehat{p}_{O_2}(O_3) + \cdots + \widehat{p}_{O_n}(O). \qquad (4)$$

The principle of the **polygon of accelerations** is thus seen to stand in complete parallel to the idea as applied to velocities. It may be formulated as follows: Any polygon, plane, or gauche, may represent a resultant acceleration and a group of its components. Either direction in any side of the polygon may represent the resultant. But the components must then be taken so as to follow consecutively round the perimeter in the contrary direction. With one given resultant, an indefinite number of such polygons can be drawn.

The polygon construction for velocity and acceleration has been established in association with change of origin; though it may equally well be based upon the triangle principle (§§ 8, 32), by applying the process of decomposition, once or repeatedly, to either or both of the original components. The

arrangement in the text is not meant to indicate an exclusive interpretation for groups consisting of many components; it aims rather to present a possible meaning for them, when the number exceeds three, the limit corresponding to the common coördinate systems.

35. The central thought of the preceding section can be viewed instructively as a direct consequence of combining the leading ideas of § 12 and § 29. The relation shown by equation (3), § 12, is permanently true. If the triangle that represents this be drawn for two epochs, each side of the triangle for the first epoch may be regarded as transformed into the corresponding side for the second epoch by the geometrical addition of a definite velocity-component. Let AB (Fig. 18) represent $v_Q(O)$; AC represent $v_Q(O_1)$; and CB represent $v_{O_1}(O)$; all at $t = 0$. Let the same quantities at $t = \Delta t$ be represented by AB', AC', $C'B'$, respectively. Then the components that must be added in order to change each velocity properly in direction and magnitude are represented by BB', $CC' = BD$, and DB', if $C'D$ be drawn equal and parallel to CB. But in the triangle BDB' thus constructed, BB' represents the resultant, BD and DB' represent components, and this relation is also permanent, whatever the interval between the two epochs may be. According to the idea of § 29, however, the time-rate and direction of BB' determine at the limit the acceleration of $v_Q(O)$, or $p_Q(O)$. BD is similarly related to $p_Q(O_1)$, and DB' to $p_{O_1}(O)$. Hence the limiting relations of the triangle BDB' yield as before,

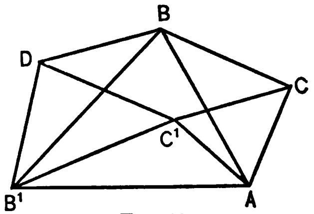


Fig. 18

$$\widehat{p}_Q(O) = \widehat{p}_Q(O_1) + \widehat{p}_{O_1}(O).$$

36. Equations (1), § 34, express the accelerations parallel to the reference-axes in their relation to the second derivatives with respect to time of coördinates parallel to the axes, but measured from a moving point. They repeat the lesson (cf. § 30, end) that the acceleration parallel to a line is not in general the second derivative of a coördinate in the same direction, even though the latter is measured from a point which is instantaneously at rest.

The directions also of coördinate lines may be variable; and we proceed to derive the values of accelerations parallel to the reference-axes, when coördinates of that character are introduced into them. The discussion will not be extended beyond the limits set in the parallel case for velocity (§ 15). Differentiating equations (2) of that section with respect to time, the values of $p_x = \frac{d^2x}{dt^2}$, and $p_y = \frac{d^2y}{dt^2}$, are obtained. The results are easily verified; but the number of terms appearing exhibits the noticeable complication due to the use of "moving coördinates," even when they are measured along perpendicular axes and confined to one plane. The values are

$$p_x = \frac{d^2x_1}{dt^2}\cos\gamma - \frac{d^2y_1}{dt^2}\sin\gamma - 2\,\omega\left(\frac{dx_1}{dt}\sin\gamma + \frac{dy_1}{dt}\cos\gamma\right)$$
$$-\frac{d\omega}{dt}(y_1\cos\gamma + x_1\sin\gamma) - \omega^2(x_1\cos\gamma - y_1\sin\gamma); \qquad (1)$$

$$p_y = \frac{d^2y_1}{dt^2}\cos\gamma + \frac{d^2x_1}{dt^2}\sin\gamma + 2\,\omega\left(\frac{dx_1}{dt}\cos\gamma - \frac{dy_1}{dt}\sin\gamma\right)$$
$$+\frac{d\omega}{dt}(x_1\cos\gamma - y_1\sin\gamma) - \omega^2(y_1\cos\gamma + x_1\sin\gamma). \qquad (2)$$

These equations satisfy the requirement of expressing the accelerations parallel to any rectangular reference-axes X and Y, and consequently the resultant acceleration also, in terms of x_1, y_1, γ, and their time-derivatives. Before leaving the

subject, however, it will be profitable to examine more closely two particular cases. First, let X_1 coincide instantaneously with X, and Y_1 with Y. For $\gamma = 0$,

$$\left.\begin{aligned} p_x &= \frac{d^2x_1}{dt^2} - 2\,\omega\frac{dy_1}{dt} - \frac{d\omega}{dt}y_1 - \omega^2 x_1, \\ p_y &= \frac{d^2y_1}{dt^2} + 2\,\omega\frac{dx_1}{dt} + \frac{d\omega}{dt}x_1 - \omega^2 y_1. \end{aligned}\right\} \tag{3}$$

In equations (3), the types of the terms introduced by the angular motion of the coördinate lines become distinctly apparent. Either of the general values (1) and (2) is seen to be the projection of both particular values in (3) upon the axis in question. In order that p_x and p_y should be equal to $\frac{d^2x_1}{dt^2}$ and $\frac{d^2y_1}{dt^2}$, respectively, for all positions and velocities of the moving point, both ω and $\frac{d\omega}{dt}$ must be zero. Therefore the coördinate lines must be *fixed*, in order to secure this simplification; it is not sufficient that their angular velocity should be constant, nor that they should be at rest instantaneously (cf. § 34).

Secondly, let the additional condition (beside $\gamma = 0$ at the epoch) be imposed, that the line X_1 shall move so as to contain the moving point always. Then $y_1 = \frac{dy_1}{dt} = \frac{d^2y_1}{dt^2} = 0$, and equations (3) become

$$p_x = \frac{d^2x_1}{dt^2} - \omega^2 x_1; \quad p_y = 2\,\omega\frac{dx_1}{dt} + \frac{d\omega}{dt}x_1. \tag{4}$$

Aside from difference in notation, these are equations (3), § 30; which are here seen to be included in the more general scheme of the present section.

37. Equations (1) and (2) deduced above may also be interpreted as affording the basis for **change in reference-axes for acceleration** (with the limitations noted in §§ 15, 34, which

include retention of the origin). If the reference-system be O with X, Y, a moving point has an acceleration parallel to X, $p_x = \frac{d^2x}{dt^2}$; if it be O with X_1, Y_1, the same point has an acceleration parallel to X_1, $p_{x_1} = \frac{d^2x_1}{dt^2}$. Similarly $p_y = \frac{d^2y}{dt^2}$; $p_{y_1} = \frac{d^2y_1}{dt^2}$. It has been remarked already that the general relations among these derivatives may be built up from the particular forms for $\gamma = 0$, shown in equations (3), § 36. For that reason they are convenient to use in making the transfer from one system to the other; and they lend themselves easily to an analysis which assigns to each term its office in changing the direction or magnitude of velocity (cf. § 31). To this instructive analysis we now proceed, starting from the relation (§ 16),

$$\widehat{v}_Q(XY) = \widehat{v}_Q(X_1Y_1) + \widehat{r\omega}.$$

Since this is permanently true, the actual accelerations, however expressed, must be competent to preserve the adjustment indicated; and our aim must be to recognize in the separate terms of equations (3), the quantities necessary to produce all the effects in detail.

The velocity $\widehat{v}_Q(XY)$ is duly changed in magnitude, and in direction relatively to the axes (XY), by the resultant of (p_x, p_y). The terms enumerated in the second members are to be shown equivalent to this resultant. The components $\frac{d^2x_1}{dt^2}$ and $\frac{d^2y_1}{dt^2}$, taken together, carry $\widehat{v}_Q(X_1Y_1)$ through its series of values, the changes in direction being reckoned relative to the axes (X_1Y_1). There is additional change in direction of this velocity, relative to (XY), due to angular motion of (X_1Y_1) at the rate ω, which accounts for the components $\left(+\omega\frac{dx_1}{dt}, -\omega\frac{dy_1}{dt}\right)$. The same angular motion in the projections of $\widehat{r\omega}$ upon (X_1Y_1) is provided for by $(-\omega^2x_1, -\omega^2y_1)$. And, finally, the changes in magnitude of those projections are

covered by the elements $\left(\frac{d}{dt}(+\omega x_1),\ \frac{d}{dt}(-\omega y_1)\right)$. When these components are all collected, the presence of every term in the equations is explained.

It may become advisable or necessary to change the reference-system entirely by abandoning both origin and reference-axes for new ones. The elements of acceleration to be added in order to make the complete transition can be obtained by combining the results of §§ 34 and 36.

38. It has been shown how **translation** and **rotation** of a **rigid solid** involve related schemes of simultaneous velocities for its points. The corresponding groups of accelerations can now be added, by way of completing that view of the two types of motion. When a rigid solid has a motion of pure translation, the simultaneous velocities of all its points are always equal; therefore, at each instant, the accelerations throughout the solid must be equal. This common acceleration is called the **acceleration of the solid.** Since the points of the solid may describe curved paths, the resultant acceleration may be such as to produce changes in both direction and speed of the velocity of the solid.

When the motion is one of pure rotation, the accelerations of the individual points must be compatible with the necessary circular paths. At any chosen point, whose distance from the (fixed) axis of rotation is r, the elements of acceleration may be specified as $r\frac{d\omega}{dt}$ in the tangent at the instant to the particular circle described by the point, and $r\omega^2$ inward along the radius. In these expressions, ω denotes the angular velocity of the solid. Its time-rate must also apply equally at all points; hence the coefficient $\frac{d\omega}{dt}$ is called the **angular acceleration of the solid.** It is likewise the time-rate of angular velocity for

all lines of the solid which lie in planes perpendicular to the axis of rotation. The expressions for velocity and acceleration in rotation involve one factor $\left(\omega \text{ or } \frac{d\omega}{dt}\right)$ belonging to the body as a whole, while the other (r) depends upon the position in the solid of the particular point taken.

Statements concerning accelerations arising from the combination of rotation with translation will be reserved until special need for those results presents itself. And the discussion is also postponed which considers the scheme of necessary accelerations when the axis of rotation is in general an instantaneous axis of variable direction.

DYNAMICAL

CHAPTER III

MASS AND FORCE

39. The science of Physics is concerned with certain properties of bodies. The ideal set before it may be roughly outlined as that of presenting an account of them which is complete, connected, and inclusive of quantitative relations. According to the particular branch of physics selected for treatment, one group or another of properties will assume especial prominence. For example, transparency, color, refractive power, are important in relation to Light; boiling-point, specific heat, involve properties to be specially studied under the heading Heat. Within the boundaries of Mechanics also, as a subdivision of physics, there is at least one such property, fundamental for our immediate purpose. It is a matter of universal experience that bodies are brought into motion and stopped, relatively to the earth's surface, with widely varying degrees of difficulty. Such differences seem to be resident in the bodies themselves, and inseparable from them. The conception of this property is gained individually, and was in all probability originally derived, in connection with muscular effort. But the differences in question persist, whether changes in motion be produced by muscular exertion, or by other

equivalent agencies. They appear, even though the effects of visible or invisible connection with the earth are eliminated as completely as possible (*e.g.* in the case of sliding on a horizontal plane, with minimized friction). They exhibit themselves, too, among bodies that are of the same shape and volume, and therefore geometrically alike.

An element derived from observed phenomena is here introduced into Mechanics, in addition to those fundamentally assumed by the abstracter sciences, Geometry and Kinematics. This element differentiates bodies among which the latter sciences do not need to distinguish. It may be regarded as affording a basis for ideas of inertia and force.

40. Bodies are said to manifest **inertia** in proportion as it is more difficult to set them in motion, the difficulty here referred to being inherent or essential, and not due to circumstances which can be identified as accidental, such as the *direction* of the motion. This furnishes one ground of discrimination between inertia and weight; for the latter increases the difficulty of producing motion in the upward vertical, while the former is associated with all directions indifferently.

For the purpose of a first explanation, the reference-system shall be one that is fixed relatively to the earth's surface. The statement of dynamical ideas will then be provisional; the discussion necessary in order to show what other groups of reference-systems the scheme of thought includes and excludes is taken up in §§ 51, 52. This procedure utilizes the narrower suggestions of established habit, and it may properly be termed natural, in so far as we are thus retracing the steps by which the conceptions of our science have been disentangled from the phenomena and formulated in the process of historical development. Further, since a motion of translation requires that velocities — and likewise accelerations — shall be simul-

taneously equal throughout a rigid body of finite extent, our primary thought can be most readily attached to that simplified case. A body is (experimentally) rigid if the deformations produced in it are negligible.

Speaking with these temporary limitations, then, **force** is said to be acting whenever the physical conditions are such that velocity is changed in magnitude or direction. When the time is shortened within which a given series of changes can be produced in the velocity of the same body, it is assumed that the force has been increased. According to the ideas introduced in Chapter II, change of velocity as related to time measures acceleration. The central importance of acceleration in the present connection justifies the prominence already given to it, and the detail with which the preliminary conceptions have been discussed.

It is easier to start a train of empty freight cars upon a level track, than to set the same cars in motion when loaded, to an extent far exceeding the change in frictional resistance.

In comparing two magnets, that one exerts a greater force upon a given piece of soft iron, which more quickly brings its speed from zero to an assigned value.

The same general result is seen in another instance, where the motion is not one of translation. Let a ball revolve uniformly about a vertical axis, to which it is attached by a horizontal rod. The velocity of the centre (which is the average in magnitude for the whole ball) is turned through the angle 2π in the time of one revolution. If, now, the distance between the centre of the ball and the axis be shortened, and the angular velocity be increased until the speed of the centre attains the same value as before, the turning through the angle 2π will occur in shorter time, and the inward pull of the rod upon the ball is found to be greater.

The starting-points in our sense-perceptions are facts at the foundation of Mechanics, and incorporated into its thought. The branch of knowledge with which we are dealing exhibits in remarkable measure, it is true, a commanding view of complicated phenomena, obtained by processes of mental analysis, that are now carried on with very few direct appeals to experiment. But the basis in experience cannot be ignored, without falling into the error of treating Mechanics as though it were a

part of pure mathematics. The original empirical data must not be overlooked, while we endeavor to appreciate the conceptions in their extended scope, for application within widened range.

41. Two sets of physical conditions are considered as calling into play forces of equal magnitude, if they are capable of producing in a given body accelerations that are numerically equal. The accelerations that are compared may be of either type, tangential or normal, since a motion of translation is consistent with both.

Two forces are also to be considered equal in magnitude, if each is capable of preventing the acceleration producible by the same third force. For instance, the downward vertical force upon two bodies can under ordinary conditions be assumed equal if, when hung at rest from the same spring-balance, they stretch it to the same reading. This latter test of equality is sometimes spoken of as **static**, the former being then characterized as **dynamic**. In connection with this example, it becomes apparent that forces (of great magnitude even) may be exerted, as evidenced by the final deformation or rupture of a body, without producing acceleration. But then the forces must neutralize each other, and are, in their sum, zero.

Force has direction as well as magnitude. Since it is commonly regarded as the cause of acceleration, or as giving quantitative expression to the physical conditions, in so far as they determine acceleration, the direction of the force is that of the acceleration which it produces.

Let a piece of iron be thought of as free to move on a smooth horizontal plane, and as initially at rest relatively to the plane. Let it be moved over the plane, first by a magnet, and then by a spring-balance, the pull of the latter being always adjusted so that the kinematical phenomena are the same for both conditions. Then the forces exercised parallel to the plane by the magnet and the spring-balance are equal at corresponding instants.

The operation of "weighing" with an equal-arm balance is essentially one of adjusting forces to equality, without otherwise measuring them. When there is "balance," the downward pull upon the bodies in both scale-pans is neutralized. The pull downward on one side produces an equal upward pull on the other scale-pan, as a result of symmetry in the balance. The downward force due to the earth is called "weight"; hence, when there is equilibrium, the weights of the bodies in the two scale-pans are equal, if we neglect the correction for buoyancy of the air. When the process is that of "weighing by substitution," two forces are adjusted to equality by making them neutralize the same third force.

42. The notion of inertia has been made quantitative and reduced to measurement, like other central conceptions of physics. The term **mass** is in general use to denote the measured inertia of bodies. The mass ratio of two bodies is defined as the inverse ratio of the accelerations producible in them by equal forces. Let m_1 and m_2 be two masses; p_1 and p_2 the accelerations produced in them by equal forces. Then $\frac{m_2}{m_1}=\frac{p_1}{p_2}$; and if m_1 be chosen as the standard or unit, the value of the ratio will be the numerical value of the mass of the other body.

The standard of mass adopted for scientific purposes is the **gram**. This standard is preserved in the form of a kilogram (the Kilogramme Prototype des Archives) whose mass is taken as one thousand grams. The same unit, or a decimal multiple of it, is also used in many technical applications of science; and it has displaced all others in countries where business is transacted according to the metric system.

The measurement of mass might be actually executed, without extraordinary experimental difficulty, by observing the accelerations produced when different bodies were set in motion by equal forces; for example, by a spring-balance drawn out to a definite mark. But the determinations of time and length that would be needed, as well as the precautions to eliminate the effects of other forces, like friction, can all be

avoided. The method by which this can be done is therefore practically preferable. It is known that the weights of equal masses are equal at the same place on the earth's surface, and in a vacuum (§ 43). Consequently, the operation of weighing, which secures directly an equality of (uncorrected) weights, can be made the means of determining equal masses, and therefore of ascertaining the mass of any body, by comparison with variable combinations of standards.

The names "inertia" and "mass," as interpreted above, are seen to apply to the same property of bodies, the former being used most often qualitatively, the latter quantitatively. This redundancy has proved a fruitful source of confusion. Since the usage prevails, however, it seems to be in the interest of clearness to recognize the overlapping of the two terms, as well as the distinction that is more or less consciously observed between them.

43. It follows, as a corollary from the ideas already presented, that, when equal accelerations are brought about in two different bodies, the force must be greater which is applied to the body that has the greater mass. For if the forces were equal, a smaller acceleration would be shown in the larger mass (§ 42); so that an excess of force must be applied to that body, in order to equalize the accelerations. It is also accepted, in our conception of force, that it increases with the acceleration exhibited by the same body (§ 40). In conformity with these two aspects of the relation — (1) when masses are different, and accelerations equal; (2) when accelerations are different, and masses equal —force is in fact measured as *proportional* to mass and acceleration "conjointly"; *i.e.* to the product of mass and acceleration. Force has the sign of its acceleration, mass being essentially positive (signless).

As in other cases where one quantity is proportional to the product of two others, the simplest numerical relations will be established, if matters are so arranged that the proportional

factor is unity. This idea being applied, after selecting as fundamental standards the second, the centimeter, and the gram, the force unit is fixed which is known as the **dyne**. One dyne of force is active, when equal unit accelerations (in terms of centimeters and seconds) are shown throughout the mass of one gram. And, in general, Force (dynes) = Mass (grams) × Acceleration $\left(\frac{\text{cm.}}{\text{sec.}^2}\right)$. This quantitative relation is, of course, equally valid, whether the acceleration be constant as regards time, or variable. In the latter case, the variable force has instantaneous magnitude and direction; in the former, these elements are both permanent, and the force is constant.

The plan of measurement whose essentials are thus determined is called the C.G.S. system, in accordance with the initials of its fundamental standards. It will be used consistently in all the chapters which follow, except Chapter IX. That is in part devoted to the necessary explanations connected with another system which still survives in common life, and in applications of Mechanics to some branches of engineering.

After any appropriate instrument has been calibrated to indicate dynes, it may be used to prevent acceleration, and measure other forces by the "zero method" of neutralizing them. A spring-balance so graduated may hold a body at rest, in opposition to the tendencies of weight, or magnetic action, and the reading of the balance will then measure those tendencies in dynes.

If two masses of m_1 and m_2 grams have motions of translation with accelerations p_1 and p_2 $\left(\frac{\text{cm.}}{\text{sec.}^2}\right)$, the corresponding forces, measured in dynes, will be $P_1 = m_1 p_1$, and $P_2 = m_2 p_2$; and therefore

$$\frac{P_1}{P_2} = \frac{m_1 p_1}{m_2 p_2}. \tag{1}$$

For small vertical ranges, and in a vacuum, all bodies falling freely show very approximately equal accelerations. Applying equation (1) to that case, and dividing by the (common) acceleration-factor, we have

$$\frac{P_1}{P_2}=\frac{m_1}{m_2}. \tag{2}$$

That is, weight is proportional to mass; and, more particularly, equal masses have equal weights. This basis for the operation of weighing, alluded to above (§ 42), was laid with as great accuracy as the circumstances then allowed, by the classic experiments of Galileo with falling bodies. In the sequel it will be pointed out how Newton's observations of pendulum motion verified the conclusion with all necessary precision (§ 101).

44. The selection of acceleration as the kinematical factor in the expression for force followed from Galileo's discussion of his own experimental results. It is on record that the experiments suggested at least two forms of the kinematical element; but upon comparison of them, one was finally rejected as leading to less simple expressions.*

Mechanics has its share in that kind of development which physical science everywhere exhibits. This consists in gaining clearer insight into phenomena; formulating conclusions with greater comprehensiveness; and adapting thought more closely to the suggestions of experiment and observation. The progress of the conceptions, Force, Acceleration, Mass, toward their present clearness and inclusiveness is distinctly traceable in the history of the science. It is clearly apparent, for instance, that the idea of mass was only slowly discerned among associated phenomena, and seen to be the unchanging element that underlies variable weight. The histories of physics quote the comparative observations with the same clock at Paris and Cayenne (Richer, 1671) as furnishing one important clew.

* Mach, "Science of Mechanics" (translation), Chapter II, Section 1, "Galileo's Achievements."

The concurrent evidence of all measurement bearing on the point shows inertia to occupy a place which is unique among physical properties, and which justifies its selection for the foundation of quantitative physical reasoning. By way of enforcing this position, we shall summarize here the important experimental results. Although familiar, they deserve the emphasis of citation.

(1) The mass ratio of two given bodies is independent of time, position, variation in accompanying properties, and the particular method of measurement.

Apparent changes with time have hitherto always been successfully traced to other sources; *e.g.* changes in buoyancy of the air, evaporation, oxidation, occlusion of gases. Apparent change with position on the earth's surface is consistently explained as due to changed intensity of the "weight-field." Nor is the independence of position contradicted by any phenomena of astronomy, when observed values are compared with calculations in which it is a fundamental assumption that mass is constant. The constancy of mass in the face of all forms of chemical action, which yield "new substances," with change of every other property (except weight), has been the corner-stone of quantitative chemistry since the epoch of Lavoisier. It is often quoted as the "Law of Conservation of Matter" (*i.e.* mass). The mass of a raindrop, or a man, varies with time; but "given body" is not to be so understood as to include such cases of growth. Methods of measuring mass, in order to be legitimate, must really exercise equal forces upon the bodies, if the mass ratio is to be the inverse acceleration ratio. Pieces of iron and brass have not masses that are in the inverse ratio of the accelerations produced in them by the same magnet. The proper sifting of evidence in such cases leads to the assumption, as the most simple one, that mass has the same value for the same body, as related to all exhibitions of force; and finally results in measuring the forces applied.

(2) All substances that are capable of affecting us through any channel of sense-perception, are found to exhibit consequences of inertia. The phenomena due to inertia may even be the sole indication that the substance is present; so that they have come to be looked upon as the final test of **matter.**

The evidence of inertia may be direct; *e.g.* resistance of wind, objects affecting our sense of touch; or it may be indirect. Thus the medium for light and electrical action must possess inertia, according to our present conceptions, in order that communication of motion may require time, and waves may travel with definite speed. In this sense, ether is one form of matter.

Mechanics then "describes the motions occurring in nature," in terms which involve the one quality of bodies that appears to be universal and invariable. It is said that "Mass is an absolute positive constant for all bodies." There is one common property here in question, which always remains, after making abstraction of all the other properties with which it occurs associated. Hence follow quite naturally the widely ramified relations of Mechanics to special sciences concerned with other and more controllable or variable properties of bodies. Moreover, since inertia is permanent in amount, and inevitable, we can easily account for the inclination to regard mass as though it were not merely a measured quality, but were the substratum of all other qualities, and gave directly the "Quantity of Matter" in a body.

45. In the sections immediately preceding, the aim has been to explain fairly in outline the scheme of ideas that underlies the conceptions of mass and force, and the adoption of standards or units for them. Such provisional treatment must, however, finally be extended, in order to cancel in part the restrictions thus far maintained. We proceed first to remove that limitation which confined the discussion to motion of translation.

If a rigid body has any motion of rotation, either pure or combined with translation, there is obviously no longer a common acceleration-factor by which to multiply the total mass of the body where it is desired to write an expression for the force. The simultaneous accelerations at the various

points will in general differ in both magnitude and direction. For wider application, therefore, the fundamental statement regarding force cannot be made of a body or system as a whole, but must be so formulated as to apply differentially to the parts of the mass. When a body or system moves with acceleration, a group of differential forces is to be considered as acting, each immediately upon its own differential mass, and in the direction of the acceleration actually existing there. Then each force will have its **point of application** (at a particular differential mass), and its **line of action** (determined by the acceleration of that differential mass). So that a force is not fully specified by its magnitude and direction, for these would assign it only to a group of parallel vectors; but in addition it is individualized in this group by connecting it with a definite element of mass, and in this way specifying the **position** of the force.

An example of such differential distribution of force is seen in gravitation, which, as conceived since Newton's time, takes hold directly upon every part of a mass. Weight, analyzed as gravitation modified by the earth's diurnal rotation, is so distributed. In another class of cases, however, the action upon a body can be traced from the outside to a limited area of its boundary, and then there is automatic adjustment of force within the body, due to the various forms of rigidity. These resist any relative displacement between parts by compression, torsion, etc., and constrain them to participate in a grouping of accelerations compatible with preserving the integrity of the body. The pull of a locomotive is thus applied to a train through the draw-bar; the action of ropes, belts, and connecting parts in machinery is quite generally of the same type; as is also that of columns and members of trusses in "static" constructions.

46. Under the supposition that mass, and consequently force also, is continuously distributed within the boundary of a body, the perpendicular distance $\bar{x}$ of its centre of mass from the YZ reference-plane is given by the equation

$$\int_0^{m_1} x\,dm = m_1\bar{x}$$

(cf. § 145 and note the extended meaning of the integral sign). If this be differentiated twice with respect to time the result is

$$\int_0^{m_1} \frac{d^2x}{dt^2}\, dm = m_1 \frac{d^2\bar{x}}{dt^2}. \tag{1}$$

Since $\frac{d^2x}{dt^2}$ is the acceleration parallel to X of dm, the integral in equation (1) represents the sum of the corresponding group of differential forces dP_x. This sum is formed by algebraic addition of the parallel force-magnitudes, without respect to their position, and it is so taken as to include all parts of the mass m_1. It will be termed the **total force** parallel to X, and written P_x; but where no confusion seems likely to arise from doing so, the word "total" can be omitted. With that notation equation (1) may be put into the form,

$$P_x = m_1 \frac{d^2\bar{x}}{dt^2}, \tag{2}$$

and applied without restriction to all kinds of motion. This expression is of the same type as that already written for translation, but it gives greater extension — and at the same time precision — to the thought that force is measured by the product of mass and acceleration. Equation (2) makes apparent the sense in which it can be said that the same basis for measurement can be retained in all cases. We mean that one effect of any distribution of force parallel to a given line is always measured by the product of the total mass and an acceleration-factor. The difference between the more special conditions and the general case is seen to be that the really *common* acceleration of the former must in the latter be replaced by an *average* factor, which is the actual acceleration of the centre of mass parallel to the line in question. It is important to notice that the magnitude of this acceleration, provided the mass is the same, depends upon the aggregate or total force

only, and not upon the positions of the elements of force entering into the sum. This does not exclude the existence of other effects of the force-distribution that are determined by the positions of the force-elements; these are reserved for subsequent discussion (cf. § 62).

47. In the extensive class of problems where constraint is imposed by what is termed rigidity, the forces upon certain elements of mass will in part at least be brought to bear internally by the adjoining portion of the body itself. Therefore, if the group of forces distributed among the various elements of mass be considered in detail, they may be classified according to their mode of application as external or internal in relation to the entire body. On the other hand, the first preliminary inspection of familiar instances inclines us to think of the actions upon a given body which are measured in terms of force, as determined by conditions external to the mass affected. This difference of view, however, implies no real difficulty or contradiction; and we shall next undertake to make clear in what respects the two notions are essentially equivalent.

It is assumed, as one fundamental postulate of Mechanics, that two masses are reciprocally concerned in all manifestations of force. By this is meant: (1) That every exhibition of force upon one mass is attributable to some other mass as its source; (2) That two masses thus related affect each other mutually, with equal and opposite forces in the same line. The conception of **stress** includes both forces of such a relation. Force appears as stress when the treatment extends to both masses concerned; or stress as force, when one of them is temporarily excluded. This postulate, like many other generalizations of theoretical science, is suggested by results that are accessible to observation, and, within certain limits, it is veri-

fiable by measurement. Its justification, when employed with broader scope, is found in the universal agreement with experience of conclusions derived from it by legitimate deduction (cf. §§ 61, 74).

A strong spring (dynamometer) inserted between a locomotive and train in motion shows equal pull backward upon the locomotive, and forward upon the train. The tension of a stretched wire, or the thrust of a beam, is exerted at both ends, in opposed directions, upon the two bodies between which it forms the visible connection. When there is an assumed invisible connection, the same kind of interaction can be detected. If a piece of iron is magnetized by induction and moved, a magnet of some type can always be indicated as giving rise to the force of attraction. Experiment can be made to show, too, that a permanent magnet and attracted iron exercise equal and opposite forces upon each other. Electrical attractions and repulsions yield results in harmony with these.

It is to be remarked about the last two instances that the source now recognized for the action may prove to be such in appearance only. The strong tendency at present is to assign an important rôle to the electromagnetic "field-medium" in the production of "ponderomotive forces" upon bodies situated in it. But the gist of the matter remains untouched by such changes of attitude; the active second body is only located elsewhere.

The conclusions drawn directly from observation have been extended in both directions as regards the magnitude of the masses involved. No contradictions present themselves when it is assumed that the earth and a stone pull upon each other with equal and opposite forces, or that the earth and the moon do so. The results of the same rule are found to hold true when it is applied to the contiguous differential parts of a body which are exerting forces one upon another in exercising constraints.

48. Return now to the relation $P_x = \int_0^{m_1} p_x\,dm$, which expresses P_x as the integral sum of all differential forces parallel to X throughout the body. It is evident that no forces can appear in a sum of this character which are exerted between parts of the total mass included. For such internal forces, by the supposition of our postulate above, must always occur in self-cancelling pairs when the integration covers the region specified. The final summation of the group of parallel forces

acting upon the differential masses will contain, as effective elements, the forces originating in masses external to the boundary of the body considered. The general conclusion is, therefore, in harmony with the almost intuitive perception that actions determinative of acceleration (in a rigid body) must proceed from conditions external to the mass affected. The body itself, indeed, modifies any result quantitatively, but only through its susceptibility to particular external influences.

49. At each differential mass the resultant acceleration can be conceived as made up of partial accelerations, the possible sets being indefinite in number, but each fulfilling the polygon condition (§ 34). The resultant may be said to contain all such elements within itself, every one counting for exactly as much in the general effect upon velocity as though it were alone. The resultant is equivalent to any coexistent set of its components, obtainable from it by orthogonal or oblique projection. This independent effect of each element, irrespective of the presence or absence of others, is a mere unfolding of consequences implied in constructing the definitions of acceleration and velocity.

For each type of acceleration, separately, there is a corresponding force. Each differential mass may give evidence of **tangential** and **normal force**, associated with its tangential and normal accelerations. Distinguishing upon another basis, we should speak of **force** as **radial**, or **perpendicular** to the **radius-vector**; and just so **resultant** and **component forces** are related to their accelerations. The distinction between component force and force parallel to a fixed line follows that established for accelerations and velocities (§§ 9, 32). The conception of total force has been confined to the type last mentioned; but when oblique reference-axes are contemplated, nothing hinders the introduction of total component force, provided that the

decomposition be systematically effected at each differential mass according to the same plan. Further, we may, when translation alone is under consideration, with equal propriety, speak of total tangential, or normal, or resultant force, because of the necessary parallelism of those elements at each part of the total mass. But similar totals cannot be formed from the polar component forces because the requisite parallelism is excluded by the scheme of decomposition.

Whichever plan is used, the total force will contain as effective elements only external action parallel to the line taken.

50. The same degree of independence prevails among force-components as among the corresponding accelerations. But such a truth about a physical quantity cannot be based upon definition merely; it must harmonize and contain the results of all observation in the field. As a résumé of all experience, we are justified in announcing that any given set of physical conditions is registered in the actual effect by a group of accelerations that is invariable for the same body, whether other conditions that can produce acceleration are present or absent. The acceleration that really occurs at each part of the body is the geometrical sum of the elements of acceleration producible there by the sets of conditions separately. In other words, the product of mass and acceleration is a definite measure, universally applicable, of this aspect of all physical conditions, under all circumstances, whatever combinations arise. This statement is sometimes introduced as the "Principle of the Independence of Forces." It is, however, scarcely a new principle, but rather an explicit recognition of the ground for adopting force as a convenient and sufficient physical concept.

As regards action upon each differential mass, therefore, forces can be combined by the constructions for the triangle,

parallelogram, and polygon, with graphical and analytical relations that follow the same lines as those for acceleration (§§ 32, 34). The difference is solely that the mass-factor is introduced everywhere, to express force instead of acceleration.

The idea that is prominent in the foregoing paragraph relates to compounding known component forces into a resultant, where physical conditions are superposed, whose measure separately would be the components. It has been shown how the resultant force is definitely calculable, when the components are given. But it must be noted that the reverse process, by which a given resultant force is resolved into partial forces, affords no sure ground for accepting as physical truth the presence of such conditions as would manifest themselves in the partial forces. The analysis suggests *possible* physical conditions only, which, if concurrent, would yield the same (external) result in motion of the body. With reference to every acceleration which the mathematical (kinematical) analysis may suggest, the pertinent question is raised: "Does this interpret the physical conditions in agreement with evidence or suggestion from other sides?" As a matter of fact, the same resultant acceleration (and force) can be produced in a variety of ways, equivalent in this respect, though differing widely in other phenomena by which the acceleration is accompanied. And, conversely, the phenomena attendant upon a resultant acceleration may be determined in part by the *coexistence* of the superposed conditions, and show important modifications as compared with the effects of the same conditions severally.

The principle of superposition is everywhere illustrated in the production of mechanical effect on a larger scale, exacter measurement corroborating the conclusions of direct perception from the general character of the results. A boat can be towed, or a stone can be hoisted, by the pull of a single rope, or by the simultaneous action of two ropes.

The forces exerted in either case are calculable as to magnitude and direction in conformity with the central idea. All typical sources of motion, like weight, machines, friction, magnetic action, when coöperating, are found related strictly in agreement with the same fundamental supposition.

For the immediate purposes of Mechanics, all physical conditions are rated as equivalent, without further inquiry, which can produce the same phenomena of motion in a body. Nevertheless it is not desirable to lose touch with the wider scope of physics, by turning attention completely away from the narrow basis of that equivalence, and suppressing all reference to the variation in other effects which may accompany an equality on this one side. For example, the tendency of confining treatment to bodies supposed rigid is toward ignoring all stresses, internal or external. But the rigid body must always be held in mind as a limiting case, while actual bodies depend upon the deformation or strain, produced by external stress or other causes, for the "transmission of force," and their adjustment to varied conditions. Though the *amount* of the strain may be negligible, the *process* is part of our physical thought.

A body at rest in a tightly screwed vise, or lying loose on a table, may present the same phenomena of motion; so may a nail when falling freely, or when forced into a board; but even the salient phenomena accompanying the motion are not the same. Again, a piece of iron may be moving vertically upward, on one occasion because it has been immersed in mercury, on another because it is attracted by a magnet. Suppose the motions are identical; yet we have in the second case, besides other differences, all the internal changes connected with magnetization. Such instances can be multiplied indefinitely.

51. Another important phase of the matter can now be approached, whose discussion will develop the physical reasons for selecting one reference-system rather than another, and thereby fixing the accelerations to which forces are proportional. From the kinematical point of view, the particular velocities and accelerations in terms of which a motion is to be described are a matter of convenience alone, all schemes of description being on a par otherwise. But force is a physical quantity whose amount is not a mere question of mathematical validity. We must learn to distinguish between statements of acceleration as kinematical (or possible) and dynamical (or

actual). From the physical point of view, the corresponding forces may be classed as apparent (fictitious) or real. The criterion for distinguishing is not difficult to state or to understand. There is a general view of the physical actions determining motion, which has been gradually secured by analyzing the phenomena, and noticing lines of convergence in the evidence presented. That view suggests a reference-system that may be characterized as "fundamental," because it affords a background for such description of dynamical results as is at the same time the simplest and the most adequate among those offered to our choice. The fundamental reference-system, to which physical thought of the present day has adjusted itself, accepts the general arrangement of the stars as being fixed, while it concedes the motion of individual stars relative to the general arrangement. For all purposes of quantitative physics and astronomy we are led to results consistent with the recognized physical actions among bodies, when we assume that measurable force is acting upon the mass which exhibits measurable acceleration referred to such a system. And the force-statements connected with any given reference-system are accounted more accurate approximations to the forces really active, in proportion as the resulting accelerations are accommodated more closely to those relative to the fundamental system (cf. § 104).

Kinematically it may be said with equal truth that the earth's centre moves in an orbit round the sun, or the sun's centre round the earth. But there are physical reasons for preferring the former expression, which are practically summarized in the remark that we obtain thus a simpler view of the dynamical phenomena presented by the solar system. If a ball be thrown vertically upward, the earth has acceleration relatively to the ball, but the magnitude of the force active is held to be essentially that which gives to the ball acceleration relative to the earth, and not *vice versa*. A reference-system may be fixed relatively to a wagon-wheel as it rolls along a road; and the accelerations of points in the earth may be expressed with reference to that system. But it has become instinc-

tive not to think of forces called into play here that would give to the mass of the earth accelerations of the amount thus calculated; and closer examination corroborates the instinctive judgment. Again, if a rigid body be in rotation about an axis fixed relatively to the earth, no part of the body has acceleration, if a reference-system be taken with origin on the rotation-axis, and reference-axes fixed in the body. Yet analysis of the phenomena leads to the conclusion that great forces may be active between parts of such a rotating body; and these fit, in their essential magnitude and distribution, the accelerations that result from choosing a reference-system fixed relatively to the stars.

The process of dissociating the ideas of rest, motion, and force from necessary connection with the earth (and still preserving their character of "relativeness") can be readily traced in connection with astronomical discovery through the period beginning with Copernicus. Not only have successive steps been taken in transferring the base to the sun and to the stars, but in doing so the mind has become familiarized with the process, so as to prepare the way for other changes of base in the future, if physical knowledge in its development should make plain any advantage to be gained thereby.

The simplicity of statement (alluded to above) sought in physical science is not necessarily such as to be apparent in each single case, but it becomes evident when matters are regarded comprehensively within wider ranges. The rectilinear propagation of light, when viewed by itself, is more simply explained by the emission theory than by the wave theory. Yet these conditions are reversed when the subject of Light is considered in its entirety. In the same way the fundamental notions of dynamics, which determine the choice of its reference-system, are cast in the form most advantageous for the science as a whole. It need not be expected that they will always lend themselves most easily to a particular application. In fact, cases do appear, and some will be discussed later, where the instinctive view is held with peculiar tenacity, although it involves the acceptance of "fictitious" forces (cf. § 52, end, § 105).

52. Any reference-system fixed relatively to the earth (earth-system), to which type attention has been confined thus far in the treatment of forces, gives accelerations (and forces) that are approximate only in comparison with the fundamental system, for both origin and axes are moving relatively to the latter, so that there are force-terms eliminated which depend upon these motions. The forces cor-

responding to the earth-system form, however, a natural starting-point for closer approximation. The values thus derived, though affected with "apparent" terms, are the normal data of experimental work, and they are sufficiently close approximations for the large majority of physical problems. In departing from these values the inclusion of additional elements of acceleration will be a change toward the forces of the fundamental system or away from them. In the first case, it is said that **real corrections** are introduced into the **force-statement.** In the second, the new forces introduced are **apparent** or **fictitious** (§ 51); that is, have no basis in any recognized mutual action between bodies.

It is not necessary in every investigation to carry reference so far back as the fundamental system. For certain purposes the corrections compared with the earth-system are negligible: (1) when their ratio to the forces actually considered is small; (2) when they apply to all masses of a problem which is concerned with differences only among active forces. Considerations of this nature render it permissible, for most practical ends, to stop short of the fundamental system, with consequent omission of (known or unknown) force-elements. In other words, limits are set in practice to the *forces to be taken account of.* By this exposition of the situation adequate reasons are assigned why the earth-system should be retained, at least as a point of departure, for the work of the following chapters. In the few instances where corrections need to be applied finally to the resulting force-statements, that fact will be explicitly noted. Our ordinary procedure amounts to neglecting all accelerations of points in the earth, and the corresponding force-elements of "higher order." When corrections do become necessary, the process of adding them is facilitated by the idea of "Independence of Forces"; the new forces simply enter into a vector sum with those previously considered.

Mathematical expressions of physical fact are, in general, approximate, because it is practically impossible to represent *every* influence in the result. It is a universal method in all departments of applied mathematics to obtain a working basis in statements founded upon partial inclusion of conditions, selection being first made of those which are quantitatively predominant. By a process of abstraction the problem is partially isolated in the world of physical action ; and the treatment is simplified by conscious suppression of certain elements. To apply corrections is to take fuller cognizance of conditions by successive steps. The primary object in view may be to keep within the scope of elementary method ; or the terms omitted may not affect numerical results within the degree of accuracy desirable or attainable. One case in point is the use of the earth-system with deliberate knowledge of its theoretic imperfection. Again, the broad lines of mechanical principles can be laid down while we observe a limitation in regarding bodies as rigid. This is equivalent to cutting out the deformations that it is the prime purpose of the "Theory of Elasticity" to bring to quantitative expression. The same idea runs through the optics of lenses ; first the main thought, then the closer correction. Or through the operations of surveying ; first the earth's surface is assumed a plane ; geodesy refines upon this by treating the earth as a sphere, and next as a spheroid.

The relation between apparent force $(\widehat{A})$, real force $(\widehat{R})$, and correction $(\widehat{C})$ may always be stated mathematically under two forms :

$$(1)\quad \widehat{R} = \widehat{A} + \widehat{C}; \qquad (2)\quad \widehat{A} = \widehat{R} + (-\widehat{C}).$$

If the intuitive view should interchange $\widehat{A}$ and $\widehat{R}$ by looking upon the former as real, the correction $\widehat{C}$ would be thought of with changed sign, according to (2). This situation is in some degree realized as regards action between the earth and bodies near it. The position of physical science is that the real force here is gravitational attraction, and the apparent force weight (in vacuum). But instinctive experience suggests weight as the real force ; and if then gravitation be introduced into the scheme of thought, the final result $\widehat{A}$ (weight) is compounded of gravitation $(\widehat{R})$ and a fictitious force $(-\widehat{C})$ that is mainly "centrifugal force." (See § 105.)

CHAPTER IV

IMPULSE, MOMENTUM, WORK. KINETIC ENERGY. THE FUNDAMENTAL EQUATIONS

53. The conditions to which force is due being maintained during finite intervals of time, there is then accumulation of effect, which is registered in mathematical form as the integral of force with regard to time. This important conception has been named the **impulse** of force.

At any instant let dP_x be the element of force parallel to a fixed line X, and associated with any differential mass. The corresponding impulse for the interval beginning at t_0 and ending at t is written $\int_{t_0}^{t} dP_x dt$. The elements in the integral have each the vector properties of dP_x as regards sign, direction, and position, the time-element being conventionally positive, and affecting magnitude alone. The integral itself will be positive or negative, according as the elements of one sign or the other preponderate; it may of course be zero as one special value. It is conceived as having direction parallel to all its elements; but it cannot in the general case have position, for those elements will be parallel and not coincident.

Applying the measure of force to this instance, we have

$$dP_x = dm\frac{dv_x}{dt}. \tag{1}$$

Consequently, $$\int_{t_0}^{t} dP_x\, dt = dm\, v_x - dm\, v_{x0}, \tag{2}$$

the velocities being the values at the time-limits. The name **momentum** is applied to products of mass and velocity such as the second member of (2) presents. It is a vector quantity, whose direction and sign are those of its second factor, mass being essentially positive (signless). But position is attributed to this vector through its connection with a definite differential of mass, which also influences the magnitude of momentum. The element of position must disappear, however, when a difference like $dmv_x - dmv_{x_0}$ is considered, unless the lines of v_x and v_{x_0} coincide. Equation (2), when interpreted as applying to any differential mass, may be read: The measure of impulse parallel to any fixed line is the change in momentum parallel to that line produced during the time-interval.

Because momentum is a vector quantity proportional to velocity, it follows the same laws of projection. Resultant momentum falls in the tangent to the path; and it may be expressed in terms of its several sets of components, just like resultant velocity. The graphical constructions with triangle, parallelogram, and polygon are valid as applied to momentum (cf. §§ 8, 14).

54. The ideas of **total impulse** and **total momentum** parallel to a fixed line are made to correspond in type with that of total force, by forming the algebraic sum of all elements of impulse and momentum parallel to that line for the entire mass, without regard to their position. Here, too, the word "total" can often be omitted without introducing ambiguity. The kind of summation in question is represented by an integration of equation (2), § 53, with regard to the mass (mass-integration). Let the total mass be m_1; then

$$\int_0^{m_1}\int_{t_0}^{t} dP_x\,dt = \int_0^{m_1} dm(v_x - v_{x_0}). \tag{1}$$

But the same factor dt is necessarily common to all simultaneous force-elements, and the first mass-integral can be at once written,

$$\int_0^{m_1}\int_{t_0}^{t} dP_x\, dt = \int_{t_0}^{t} P_x\, dt. \tag{2}$$

Further,
$$\int_0^{m_1} dm\,(v_x - v_{x_0}) = m_1(\bar{v}_x - \bar{v}_{x_0}), \tag{3}$$

as a result of differentiating equation (1), § 145, once with respect to time. Equation (1) may then be put into the form,

$$\int_{t_0}^{t} P_x\, dt = m_1(\bar{v}_x - \bar{v}_{x_0}). \tag{4}$$

When the force P_x is constant its impulse is $P_x(t - t_0)$. If the body be rigid, and the motion translation, $\bar{v}_x$ and $\bar{v}_{x_0}$ are common to the whole mass. In connection with these simplified conditions, expressed in the equation,

$$P_x(t - t_0) = m_1(v_x - v_{x_0}), \tag{5}$$

the standards for numerical measurement of impulse and momentum can be defined; neither has received any special designation. There is unit impulse when a force of one dyne acts during one second. There is unit momentum when all parts of one gram mass have equal velocities of one centimeter per second. These are equivalent specifications of the units, for unit impulse produces unit momentum, starting from rest.

55. The vector character of impulse and momentum has been made prominent by introducing them in relation to a fixed line, so that summation with respect to either time or mass extends to a group of parallel elements. Some expressions for acceleration and force, however, have been formulated in connection with lines that are continuously different as the motion proceeds, though systematically chosen (§§ 26, 30). There must be an impulse for a force of that type, at least

mathematically and in a somewhat modified sense, because duration in time is common to all forces. Any such impulse is the time-integral of the general expression for the force. The differential elements in this form of impulse may have different directions; whatever coherence there is of the elements under one concept for useful interpretation is determined chiefly by their magnitude. The most important example is tangential force, for which the expression is

$$dP_t = dm\frac{dv}{dt}; \text{ its impulse being} \int_{t_0}^{t} dP_t\, dt = dm\,(v - v_0). \quad (1)$$

Here dmv is the resultant momentum at any instant; and the second member of (1) expresses the change in magnitude of that quantity, though no single direction can be assigned to the left-hand member in the general case. This association with magnitude, exclusive of direction, begins in tangential acceleration, and is continued into the quantities force and impulse derived from it. Compare this with the parallel relation between speed and velocity. Without special supposition there is no "total tangential force" in the sense defined (§ 46); and the mass-integral of equation (1) has no real meaning for our purpose unless the motion is translation. For this latter case we may write,

$$\int_{t_0}^{t} P_t\, dt = m_1(v - v_0). \quad (2)$$

56. When there is actual displacement of a body or system, while it is under the influence of forces, the effects are in general accumulated progressively as the displacement occurs. The operation of integrating force, with respect to distance passed over in its own line, embodies the idea of this summation; and the name applied to such an integral is **work**. In parallel with § 53 the work of dP_x can be written $\int_{x_0}^{x} dP_x\, dx$,

the limits being coördinate values that mark the range of the displacement. Where dP_x and dx have the same sign, that element of the integral is positive; elements are negative for which force and displacement have opposite signs. The aggregate that constitutes the integral may then be positive, negative, or zero, like any other algebraic sum. The sign of work does not depend upon the arbitrary choice of positive direction in X, because a reversal of sign in both factors does not affect their product. Moreover, since both factors are vectors, there is no ground for connecting the magnitude of work with the direction of one of them. This consideration countenances the inclusion of work among scalar quantities, which is also supported by positive and conclusive reasons to be assigned presently (§ 58). Work is indeed associated with a specified force, but merely as a positive or negative magnitude. Speaking somewhat figuratively, a force is said to "do work" when the body "yields" to it. Negative work is frequently said to be "done against the force," which is then classed as a "resistance" to the actual motion. For the purposes of general statement, force will be considered to do work in either case.

57. For any differential mass dm we may write,

$$dP_x = dm\,\frac{v_x\,dv_x}{dx}, \tag{1}$$

that form of the acceleration being chosen for convenience, in which the coördinate appears, and not the time. Integration with respect to x for the interval (x_0, v_{x_0}) to (x, v_x) gives

$$\int_{x_0}^{x} dP_x\,dx = \frac{dm\,v_x^2}{2} - \frac{dm\,v_{x_0}^2}{2}, \tag{2}$$

and affords a measure of the work expressed by the first member, in terms of mass and the speed at the boundaries of the interval. The product of mass by the half square of speed,

which is characteristic of such expressions, is called **kinetic energy.** It is plain that this quantity is positive; and that such factors as v_x are used in it without regard to direction or even sign, except that the speed must be taken in the line of the force for which the work is calculated. With that understanding equation (2) may be read: The work of force is measured by the change produced in the corresponding kinetic energy during the interval.

Work not being a directed quantity, the summation offers no peculiar features where the forces are of systematically varying direction, unless the acceleration is of different type from $\frac{v_x dv_x}{dx}$, and on that account the integral of the second member does not conform to the model of equation (2). Take the "radial" force as an example of less simple result (§ 30):

$$dP_r = dm\left(\frac{v_r d\,v_r}{dr} - r\omega^2\right),$$

its work being

$$\int_{r_0}^{r} dP_r\,dr = dm\,\frac{v_r^{\,2} - v_{r_0}^{\,2}}{2} - \int_{r_0}^{r} dm\,r\omega^2\,dr. \qquad (3)$$

Additional data must be available before the last integral can be reduced.

A noticeable point of difference between impulse and work is that the latter may be zero for a force which is always perpendicular to the actual displacement (cf. § 55).

58. The tangential force active can be given for any differential mass as

$$dP_t = dm\,\frac{v\,dv}{ds},$$

the work of this force for the interval (s_0, v_0) to (s, v) being then

$$\int_{s_0}^{s} dP_t\,ds = dm\left(\frac{v^2}{2} - \frac{v_0^2}{2}\right). \qquad (1)$$

Let dR be the corresponding element of resultant force, du the differential displacement in its line. If the angle between dR and dP_t is β, this will be the angle between du and ds also. Then $dP_t = dR \cos \beta$, because dR is resultant force; and $ds \cos \beta = du$, because ds is resultant element of displacement. Hence $dP_t \, ds = dR \, du$ always, and finally

$$\int_{s_0}^{s} dP_t \, ds = \int_{u_0}^{u} dR \, du. \tag{2}$$

Again, if dP_x, dP_y, dP_z, are forces parallel to the reference-axes, and acting upon the same differential mass,

$$\int_{x_0}^{x} dP_x \, dx = dm \frac{v_x^2 - v_{x_0}^2}{2}; \qquad \int_{y_0}^{y} dP_y \, dy = dm \frac{v_y^2 - v_{y_0}^2}{2};$$

$$\int_{z_0}^{z} dP_z \, dz = dm \frac{v_z^2 - v_{z_0}^2}{2}. \tag{3}$$

Adding these equations, member to member, the resulting equation is

$$\int_{x_0}^{x} dP_x \, dx + \int_{y_0}^{y} dP_y \, dy + \int_{z_0}^{z} dP_z \, dz = dm \frac{v^2 - v_0^2}{2}. \tag{4}$$

Consequently,

$$\int_{x_0}^{x} dP_x \, dx + \int_{y_0}^{y} dP_y \, dy + \int_{z_0}^{z} dP_z \, dz = \int_{s_0}^{s} dP_t \, ds = \int_{u_0}^{u} dR \, du. \tag{5}$$

Equation (5) holds true for any fixed rectangular system XYZ. In fact, $\int_{s_0}^{s} dP_t \, ds$ can be regarded as one term of a sum (cf. §§ 24, 25), of which the two other terms are zero, because force-components do no work, if taken always in a plane perpendicular to the instantaneous direction of ds.

From the above development two conclusions can be drawn. (1) The work of the resultant force is the algebraic (or magnitude) sum of the amounts of work calculated for the forces parallel to any three fixed rectangular axes. Work therefore

follows the law for scalar quantities, and not that of geometric addition, which is characteristic of vectors. In other words, there is a certain objectiveness about work; its amount is definite, and independent of how axes and signs are chosen. (2) In so far as the amount of work is concerned, the tangential force may be substituted legitimately for the resultant force.

When the operation of vector multiplication is familiar, work can be at once recognized as the scalar part in the product of $\widehat{dR}$ and $\widehat{ds}$, vectors having direction-cosines (l_1, m_1, n_1) and (l_2, m_2, n_2). For

$$\begin{aligned} dR\,du = dR\,ds\cos\beta &= dR\,ds\,(l_1l_2 + m_1m_2 + n_1n_2) \\ &= dP_x\,dx + dP_y\,dy + dP_z\,dz. \end{aligned}$$

59. The **total work** of the forces parallel to X can be expressed as the mass-integral,

$$\int_0^{m_1}\int_{x_0}^{x} dP_x\,dx = \int_0^{m_1} dm\,\frac{v_x^2 - v_{x_0}^2}{2}. \tag{1}$$

In order to reduce either of these integrals, the law of distribution through the mass must be known for the quantities involved, since it is not necessary that dx should have a common value at all parts simultaneously (cf. § 54). Under the conditions of translation, the requisite laws are given in simplest form; the displacement and the speed have each a common value. If in addition P_x is constant, equation (1) becomes

$$P_x(x - x_0) = m_1\frac{v_x^2 - v_{x_0}^2}{2}. \tag{2}$$

The standard for the numerical specification of work follows naturally from this result. If a force of one dyne is exerted uniformly while a displacement of one centimeter in its direction occurs, one (positive) unit of work is done by the force. The name adopted for this work-unit is **erg**.

If the unit for kinetic energy were expressed strictly according to its own standard conditions, we should say these are fulfilled when equal velocities of one centimeter per second exist in all parts of two grams of mass. But this would measure the effect of one erg of work, starting from rest (cf. § 54, end); and the close connection habitually recognized between work and kinetic energy favors the practice of measuring the latter at once by its equivalent in ergs.

The like simplification in mathematical statement for translation and constant force is possible when tangential forces are in question. Equation (2) then assumes the form,

$$P_t\,(s - s_0) = m_1 \frac{v^2 - v_0^2}{2}. \tag{3}$$

The latter equation can be used in defining the erg, provided it be understood that the centimeter may if necessary be measured along a curved path.

Energy is attributed to bodies and systems under various physical conditions, which is determined in amount by the work that has been done, or that will be done, within specified limits, and according to definite suppositions. The word "kinetic" suggests that such conditions are associated with motion in the present instance. From one side, the kinetic energy of a body or system represents the positive work of some forces in bringing its parts from a state of rest to their given speeds. From another point of view it measures the negative work that must be done by some forces or other, before those speeds are reduced to zero from their initial values. The latter form of the thought is expressed somewhat loosely in the current statement: "Kinetic energy is the capacity of a body to do work in virtue of its motion." Of course, in the stricter use of the conception work, its direct association is with forces, and not with bodies.

60. In the relations thus far developed, three quantities of primary importance have appeared, which are, Force, Force-impulse, Work. For each of them an equation of one common type has been constructed, which brings into connection the

quantity symbolized more directly according to its definition, and its measure as otherwise expressed. These are to be rated as fundamental equations of our dynamics; so they are brought together below, with the names by which reference will hereafter be made to them, and some comment upon their scope and use. On tracing the derivation of these equations, it is seen that all three represent a stage at which no assumption has been made, equivalent to treating a body or system as rigid, or in any way particularizing the character of the motion. Although they will be restricted to a rigid body and certain types of motion, in our present use of them, the equations themselves in the forms written are of general application; the limiting conditions will be explicitly introduced into them.

(I) $P_x(e) = m_1 \dfrac{d^2\bar{x}}{dt^2}$. Equation of motion (or force).

(II) $\displaystyle\int_{t_0}^{t} P_x(e)\,dt = m_1(\bar{v}_x - \bar{v}_{x_0})$. Impulse equation.

(III) $\displaystyle\int_0^{m_1}\int_{x_0}^{x} dP_x\,dx = \int_0^{m_1} dm\frac{v_x^2 - v_{x_0}^2}{2}$. Work equation.

The notation is intended to indicate that external forces alone appear in equations (I) and (II). This has been discussed sufficiently as regards force (§ 48); and a like conclusion must be valid in (II), for the total impulse is the impulse of the total force (§ 54). Internal forces, however, may appear in the work equation, as contributing to the total work, if any masses of the group included are capable of displacement relatively to each other. Under such conditions, both force and displacement may have opposite signs at the extremities of a line of stress, and thus add work-elements of the same sign. Consider, for example, two parts of a body joined by a stretched or compressed spring.

It must be noted that the classification of forces as external or internal is arbitrary, to the extent that it depends upon the boundaries assumed for the body or system, when the equation of motion is written. Such an equation may include a larger mass, or any selected parts of it; and forces internal to an entire body or system may be external as regards a given part. The word "total" is equally arbitrary, as applied to quantities in the fundamental equations. The "total mass" is the mass that the equations are intended to cover; then consistent limits are drawn for total force, impulse, etc. By taking such an instance as a train, and putting down the equation of motion for each car separately, it is seen that we have a group of simultaneous equations for the parts, whose sum is the equation of motion for the whole system. From the single resulting equation, forces internal to the wider boundary disappear by cancellation, though they are essential in the equations applying to parts of the system (*i.e.* the action across each section consists of stresses).

61. The importance of the equations of motion and impulse justifies their full interpretation in words.

I. The total force parallel to any fixed line contains as effective elements external forces only, parallel to that line. Their algebraic sum is measured by the product of the total mass and the instantaneous acceleration (parallel to the same line) of its centre of mass.

The resultant acceleration of the centre of mass is therefore completely determined by forming three equations of motion in which the external forces are projected upon the reference-axes. And the content of three such equations is frequently summarized in the statement: The centre of mass of any system moves as though the total mass were concentrated there, and acted upon by the external forces transferred to that point unchanged except in their position. The supposed collection of finite mass and finite force at a point is of course a mathematical fiction; but it finds especially convenient application where it is sufficient to trace the quantities connected with the motion of the centre of mass, to the neglect of other details. The body is then treated as a **particle**, meaning by that term a

finite mass of negligible dimensions. And for treatment as a particle, the centre of mass is the "representative point" at which the finite mass is conceived to be situated, the equations of motion being formed as suggested above.

II. The total impulse parallel to any fixed line is due to the external forces parallel to that line; and is measured by the product of the total mass and the change in velocity (parallel to the same line) produced at its centre of mass during the interval.

Two corollaries of this second statement deserve particular mention. First, a total impulse is zero which leaves the corresponding velocity of the centre of mass unaffected, no matter how the motion may proceed in other respects. Secondly, the total momentum is zero for all directions if the centre of mass does not move, although there may be motion relative to the centre of mass as origin; for example, if the centre of mass lies on a fixed axis about which a rigid body is rotating. The momentum of the body and that of the equivalent particle are equal for all directions.

In the reasoning of these sections, the conclusion that internal forces do not affect total force and impulse is based on the postulate of § 47. But the fact should not be overlooked that the postulate itself has been first inferred and then confirmed, by observing the ineffectiveness of mutual actions between bodies to produce motion of their common centre of mass. Any such general principle is originally an induction, whose final justification is to be sought in the universal agreement of its consequences with observed phenomena. One characteristic of Mechanics to be aware of is that its fundamental generalizations have been so extensively tested and found true, that they assume a misleading likeness to *mathematical* axioms.

62. The difference in effect that is due to the position of forces enters into the dynamical statement first through the work equation. In a given motion of a given group of masses, the displacement in the line of a force, and therefore the work,

will in general vary with the situation of the mass-element upon which the force acts. Neither work nor kinetic energy involves direct reference to the centre of mass, as force and momentum do in the equations already discussed; but the work equation can be recast so as to introduce such explicit reference. By doing this we shall make apparent another important property of the centre of mass, which enables us to state the third fundamental equation in parallel terms with the two others.

Let v be the velocity of any differential mass dm, $\bar{v}$ the velocity of the centre of mass of the body or system, and v' a component velocity of dm so chosen that $\hat{v} = \hat{\bar{v}} + \hat{v}'$. Then v' is the velocity relative to the centre of mass as origin, without change of reference-directions (§ 12); we shall speak of this as "velocity relative to the centre of mass." But $v^2 = \bar{v}^2 + v'^2 + 2\,\bar{v}v' \cos\alpha$, if α is the angle between $\bar{v}$ and v'; and the kinetic energy due to the resultant velocity of dm is

$$dm\frac{v^2}{2} = dm\,\frac{\bar{v}^2}{2} + dm\,\frac{v'^2}{2} + dm\,\bar{v}v' \cos\alpha. \tag{1}$$

The total kinetic energy due to the resultant velocities everywhere in the body or system is then the scalar sum of all such essentially positive elements, which can be written,

$$\int_0^{m_1} dm\frac{v^2}{2} = m_1\frac{\bar{v}^2}{2} + \int_0^{m_1} dm\frac{v'^2}{2} + \bar{v}\int_0^{m_1} dm\, v' \cos\alpha. \tag{2}$$

The total momentum parallel to the fixed line with which $\bar{v}$ coincides at the instant is $\int_0^{m_1} dm\,(\bar{v} + v' \cos\alpha)$, which is equal to $m_1\bar{v}$ (§§ 54, 61). Consequently $\int_0^{m_1} dm\, v' \cos\alpha = 0$, the last integral in equation (2) vanishes, and the total kinetic energy due to the resultant velocities is

$$m_1\frac{\bar{v}^2}{2} + \int_0^{m_1} dm\,\frac{v'^2}{2}.$$

Similarly, for velocity parallel to any fixed line X, the total kinetic energy is $m_1 \frac{\bar{v}_x^2}{2} + \int_0^{m_1} dm \frac{v'^2_x}{2}$, v'_x being the projection of v' upon X. Therefore, in connection with either resultant velocity or velocity parallel to a fixed line, the actual kinetic energy of any system of masses can be regarded as consisting of two parts: first, energy calculable as that of a motion where every part has equal velocity with the centre of mass; secondly, energy due to velocities everywhere relative to the centre of mass. The first part disappears when the centre of mass is at rest, like total momentum; the second is independent of the rest or motion of that point, and persists, even though it becomes fixed. The two parts being independent, they may be considered separately.

63. A like analysis is possible of the first member of the work equation. Separate the actual displacement dx parallel to X of any differential mass into two parts, one of which is the displacement of the centre of mass, the remainder being due to v'. Write

$$dx = d\bar{x} + dx',$$

and consequently,

$$\int_0^{m_1}\int_{x_0}^{x} dP_x\, dx = \int_{\bar{x}_0}^{\bar{x}} P_x\, d\bar{x} + \int_0^{m_1}\int_{x'_0}^{x'} dP_x\, dx'. \qquad (1)$$

And, using the result proved in the preceding section, the equation follows:

$$\int_{\bar{x}_0}^{\bar{x}} P_x\, d\bar{x} + \int_0^{m_1}\int_{x'_0}^{x'} dP_x\, dx' = m_1 \frac{\bar{v}_x^2 - \bar{v}_{x_0}^2}{2} + \int_0^{m_1} dm \frac{v'^2_x - v'^2_{x_0}}{2}. \quad (2)$$

Taking the value of P_x from the equation of motion, in the form

$$P_x = m_1 \frac{\bar{v}_x\, d\bar{v}_x}{d\bar{x}},$$

it is seen that the first integral in (2) and the first term of the second member are equal. Therefore equation (2) may be divided finally into two independent statements:

$$\int_{\bar{x}_0}^{\bar{x}} P_x\, d\bar{x} = m_1 \frac{\bar{v}_x^2 - \bar{v}_{x_0}^2}{2}; \tag{3}$$

and $$\int_0^{m_1}\int_{x'_0}^{x'} dP_x\, dx' = \int_0^{m_1} dm \frac{v'^2_x - v'^2_{x_0}}{2}. \tag{4}$$

Since equations of type like (3) and (4) can be constructed for all three reference-axes, the fact is now completely exhibited that any motion can be conceived, for the purpose of describing it, as derived by the superposition of two simultaneous and independent processes. First, a motion of the entire mass with its centre of mass, determined in all its elements by the equation of motion, the impulse equation, and the work equation (3) above. Secondly, a motion relative to the centre of mass, subject to the work equation (4), but not represented in the remaining equations. When the mass is treated as a particle, the second part of the motion is disregarded, and only the motion with the centre of mass is considered.

Each fundamental equation embodies a quantitative relation that affords the possibility of calculating any one of the elements involved, when all but that one are either assumed, or known by observation and experiment. The equations are distinguished from mere identities by this possibility of experimental attack from different sides. The ways in which they can be used are as varied as the forms in which the data can offer themselves. Without attempting a minute classification, it is useful to remark that the dynamical quantities, force, impulse, and work, can be calculated, when kinematical quantities like velocity and acceleration are known for a given mass; or the dynamical quantities may become known and put us in a position to calculate the kinematical elements. Experiments may have for their object to discover the value of the "field-constant," in the case of weight, magnetism, or electricity. If this is known beforehand, however, the forces acting upon a given body are known; and this leads to a knowledge of impulse and work. In the ordinary use

of the ballistic galvanometer, the work done upon the magnet is measured by a process which amounts to determining the kinetic energy imparted to it. The mechanical equivalent of heat brings to comparison a heat-quantity and work; the latter being given as known force acting through known distance. The phenomena presented by capillary waves give a basis for calculating surface-tension; or known surface-tension may predetermine the kinematical phenomena.

64. The equations of motion and impulse prove the centre of mass to be a "centre" of force and momentum also, in the sense that if the conditions prevailing at that point as regards acceleration and velocity were common to the entire mass, the total force and momentum would retain the values given by the actual distribution. But those equations are independent of the position of force and momentum; and connected with that as a consequence is the fact that neither equation is competent to furnish a detailed description of accelerations or velocities at different parts of a body or system, where we are concerned, not only with average values, but also with local departures from them. In some ways certainly it is a convenience or advantage that the divergencies from the average values (at the centre of mass) are eliminated from the results; but the same circumstance operates in other respects as a limitation. In order to include both aspects of the matter, the general nature of the quantities that are absent from equations of motion like $P_x = m_1 \frac{d^2\bar{x}}{dt^2}$ shall be discussed more at length.

Because P_x is an algebraic sum, it follows that in forming it negative quantity has offset positive quantity, so far as that is possible. Expressed in other words, P_x represents an effective sum, from which all neutralized elements of force have dropped out, by pairing off equal magnitudes of opposite sign. Such forces may constitute an internal or external stress; or, since position does not enter here, they may act in parallel

lines. This latter arrangement can be formed frequently enough, by grouping the given forces, to be considered a standard combination and named as such; the term **couple** designates two forces, equal in magnitude and parallel, but oppositely directed. The forces constituting a couple may be external or internal; their components parallel to any line X will evidently form a couple also. The force-elements removed by cancellation from P_x, therefore, consist of all stresses and couples that can be formed among the forces, whether their source is external or internal to the mass under consideration.

The various types of action that do not appear in the equation of motion are of familiar occurrence; it is sufficient to indicate a few illustrative examples. For a system comprising the earth and the moon, the attractions are internal stresses. In preparing to shoot an arrow, it is predominantly an external stress by which the bow is bent. An external couple is exemplified by a cylinder set spinning about its geometrical axis by means of tangential strings oppositely pulled at the ends of a diameter. The actions between the poles are essentially internal couples in a system composed of two ordinary steel magnets, if they are pivoted on a pin that bisects both magnetic axes. The centre of mass of the system is not accelerated by any of these couples or stresses; but various effects may be produced by them which do need to be taken account of, because they influence the actual motion, even of rigid bodies; this will be more fully seen later.

65. In the second member of the equation of motion, contributions of force from the differential masses enter in general at something less or greater than their true value into the sum

$$(dm' + dm'' + dm''' + \text{etc.})\frac{d^2\bar{x}}{dt^2} = m_1\frac{d^2\bar{x}}{dt^2},$$

according as their accelerations are really above or below the average $\left(\frac{d^2\bar{x}}{dt^2}\right)$. It lies in the nature of an average, however, that the allowances for excess and defect in comparison with it must be equal on the whole, as well as opposite; and this

leads to the conclusion that those parts of the forces $dm\frac{d^2x}{dt^2}$ which are not represented in the sum $m_1\frac{d^2\bar{x}}{dt^2}$ must be balanced against each other. The two aspects of a stress yield a couple when projected parallel to any line other than their own, each at its proper point of application. So that the couple may be made the more general idea, under which stress falls as a particular case when the lines of the parallel forces coincide; and we may say that the unrepresented forces are paired in couples. The origin of such couples in the mathematics of the matter is next to be made plain.

Assume in any body or system two elements of mass dm and dm_1, their coördinates relative to the centre of mass (§ 62) being x' and x'_1. Their coördinates relative to the reference-system being $x = \bar{x} + x'$, and $x_1 = \bar{x} + x'_1$, the forces are

$$dP_x = dm\left(\frac{d^2\bar{x}}{dt^2} + \frac{d^2x'}{dt^2}\right); \tag{1}$$

and

$$dP_{x_1} = dm_1\left(\frac{d^2\bar{x}}{dt^2} + \frac{d^2x'_1}{dt^2}\right). \tag{2}$$

These forces are included in the sum $m_1\frac{d^2\bar{x}}{dt^2}$ to the extent respectively of $dm\frac{d^2\bar{x}}{dt^2}$ and $dm_1\frac{d^2\bar{x}}{dt^2}$. This will not lead to algebraic error if dm and dm_1 be so chosen that

$$dm\frac{d^2x'}{dt^2} + dm_1\frac{d^2x'_1}{dt^2} = 0. \tag{3}$$

But as a consequence of the definition of centre of mass (cf. § 46) such pairs can always be formed, and they exhaust the entire mass. Evidently no exception needs to be made for the particular possibility $\frac{d^2x'}{dt^2} = 0$. Equation (3) shows that the parts which disappear constitute couples, when regarded as forces; and as kinematical elements in describing the

motion, they contain accelerations relative to a system with the original reference-axes, but with origin at the centre of mass. We shall speak of these in the present connection, for brevity, as forces and accelerations "relative to the centre of mass." The mass-integral of the supplementary parts is necessarily zero, but they may nevertheless produce important physical results. It is instructive, therefore, denoting the aggregate of the self-cancelling force-elements by P'_x, to introduce them and the measure of their effects into the two members of the equation of motion, which when thus completed takes the form

$$P_x + P'_x = m_1 \frac{d^2\bar{x}}{dt^2} + \int_0^{m_1} dm \frac{d^2x'}{dt^2}. \tag{4}$$

As here understood P_x contains the force-elements effective in producing motion of the body or system with its centre of mass — motion of the mass "as a whole." P'_x sums those items which exhibit themselves (if at all) in producing acceleration relative to the centre of mass; the closer analysis of their effects is to be taken up in connection with special types of motion.

Equation (4) may be separated into two independent equations of motion (cf. § 63 (2)):

$$P_x = m_1 \frac{d^2\bar{x}}{dt^2} \quad (5); \qquad \text{and } P'_x = \int_0^{m_1} dm \frac{d^2x'}{dt^2}. \tag{6}$$

Similar equations being written for all three reference-axes, they enable us to trace the motions with the centre of mass and relative to it. The actual phenomena of the motion are reproduced by the superposition of the simultaneous elements furnished by the two groups of equations.

It must be insisted upon that this method of treatment is a mode of description only, by which nothing in the physical conditions can be really modified. It is a formal consequence of the average coördinates,

velocity, and acceleration, of the centre of mass. But unless the scheme is recognized as resting upon that foundation, it may prove rather a hindrance than a help to understanding. It becomes more artificial, and takes on more nearly the character of a mathematical device, in proportion as physical connections are lacking among the group of masses; for inclusion in a system may be entirely a formal and arbitrary matter. As an example, take the case of a group of blocks, scattered at random on a table, which are without (measurable) action upon each other, but which may, if we please, be comprehended in one system. While the blocks lie at rest, let real force (due to assignable physical conditions) be applied to one block of mass m', producing translation in it. Then the centre of mass of the group will shift, $\frac{d^2\bar{x}}{dt^2}$ will assume a finite value, and m_1, the entire mass of the group, will appear in equation (5) as affected with that acceleration. The integral in equation (6) will also consist of finite terms; and there is opportunity for a mystical notion that force has been made to act upon the entire system in all its parts, although as a physical fact force has acted upon the part m' only. The rest is mathematical or apparent force, as becomes evident by writing the partial equations of motion for the masses m' and $m_1 - m'$.

$$P_x = m'\frac{d^2x}{dt^2} = m'\left(\frac{d^2\bar{x}}{dt^2} + \frac{d^2x'}{dt^2}\right). \qquad (7) \quad [\text{Translation of } m'.]$$

$$0 = (m_1 - m')\left(\frac{d^2\bar{x}}{dt^2} + \frac{d^2x''}{dt^2}\right). \qquad (8) \quad \left[m_1 - m' \text{ left at rest; } \frac{d^2\bar{x}}{dt^2} = -\frac{d^2x''}{dt^2}.\right]$$

$$P_x = m_1\frac{d^2\bar{x}}{dt^2} + m'\frac{d^2x'}{dt^2} + (m_1 - m')\frac{d^2x''}{dt^2}. \qquad (9)$$

The two apparent accelerations $\frac{d^2\bar{x}}{dt^2}$ and $\frac{d^2x''}{dt^2}$ are equal and opposite, so that the corresponding forces for the mass $m_1 - m'$ give zero sum, and may be introduced into (9) without disturbing its mathematical validity. If, however, we consider a rigid body as the other extreme case, the rigidity distributes automatically the force applied to any part; the result for a differential mass "embedded" in the body is in general to apply forces of constraint wherever the mass may be situated; and in a much truer sense (physically) there is motion of the body as a whole, accompanied by motion relative to the centre of mass.

66. In § 63, the total work has been separated into two parts by dividing the actual displacement dx into $d\bar{x}$ and dx'.

But the work may also be subdivided into the work of the "effective forces" P_x, and that of the "forces relative to the centre of mass" which are represented in $\int_0^{m_1} dm \frac{d^2x'}{dt^2}$. The total work of the forces parallel to X can be put in the form

$$\int_0^{m_1}\int_{x_0}^{x} dm \frac{d^2x}{dt^2} dx = \int_0^{m_1}\int_{x_0}^{x} dm \frac{d^2\bar{x}}{dt^2} dx + \int_0^{m_1}\int_{x_0}^{x} dm \frac{d^2x'}{dt^2} dx. \quad (1)$$

The second member indicates a subdivision on the basis named. If now in addition we use the relation $dx = d\bar{x} + dx'$, equation (1) becomes in the second member

$$\int_0^{m_1}\int_{\bar{x}_0}^{\bar{x}} dm \frac{d^2\bar{x}}{dt^2} d\bar{x} + \int_0^{m_1}\int_{x'_0}^{x'} dm \frac{d^2x'}{dt^2} dx' + \int_0^{m_1}\int_{\bar{x}_0}^{\bar{x}} dm \frac{d^2x'}{dt^2} d\bar{x}$$
$$+ \int_0^{m_1}\int_{x'_0}^{x'} dm \frac{d^2\bar{x}}{dt^2} dx'.$$

But the last two integrals vanish; the third because

$$\int_0^{m_1} dm \frac{d^2x'}{dt^2} = 0,$$

for every value of $d\bar{x}$; and the fourth because $\int_0^{m_1} dm\, dx' = 0$, for every value of $\frac{d^2\bar{x}}{dt^2}$. The first integral is

$$\int_{\bar{x}_0}^{\bar{x}} P_x\, d\bar{x} = m_1 \frac{\bar{v}_x^2 - \bar{v}_{x_0}^2}{2}. \quad (2)$$

And the second integral is

$$\int_0^{m_1}\int_{x'_0}^{x'} dP'_x\, dx' = \int_0^{m_1} dm \frac{v'^2_x - v'^2_{x_0}}{2}. \quad (3)$$

On comparing these results with those of § 63, they are seen to add another element to the view there obtained. Not only can we treat separately the motions with the centre of mass

and relative to that point by means of equations (3) and (4), § 63, but it is now seen that each part of the work can be regarded as done by its own group of forces. For displacement with the centre of mass, the work is entirely that of forces comprised in the effective sum P_x. And the work during displacement relative to the centre of mass is entirely that of forces comprised in P'_x.

CHAPTER V

UNIPLANAR MOTION. EQUIVALENT FORCE-GROUPS RESULTANTS AND EQUILIBRIUM

67. The aim of the two preceding chapters is to introduce, and present in systematic connection, the conceptions by means of which motion is described quantitatively. Our next undertaking is to study the application of those conceptions to three types of motion in a rigid body that are here to be examined more in detail: (1) Translation; (2) Rotation about a fixed axis (Pure Rotation); (3) Uniplanar motion.

Of these translation is the simplest, and in one sense the most fundamental, because, as we have seen, a convenient analysis of the general case of motion in a body or system sets off as one part a translation with its centre of mass. It can now be appreciated that it is the absence of all motion relative to that point which yields the simplifications characteristic of pure translation. Every *element*, then, vanishes in such integrals as

$$\int_0^{m_1} dm\frac{d^2x'}{dt^2},\quad \int_0^{m_1} dm\, v',\quad \int_0^{m_1} dm\frac{v'^2}{2} \qquad (\S\S\ 62,\ 63).$$

At various points in the preceding treatment the fundamental equations have been incidentally reduced to forms applying to translation, in order to define standards (§§ 43, 54, 59). Further discussion of these forms is scarcely needed at this point; but the equations that are essential are gathered together here as a first step toward a survey which will include also the two other types of motion named above.

Pure Translation

Equations of motion:

$$\text{(I)}\quad P_x = m_1\frac{d^2x}{dt^2}; \qquad \text{(II)}\quad P_t = m_1\frac{d^2s}{dt^2};$$

$$\text{(III)}\quad P_n = -m_1 v\frac{d\alpha}{dt} = -m_1\frac{v^2}{\rho}.$$

Impulse equations:

$$\text{(IV)}\quad \int_{t_0}^{t} P_x\,dt = m_1(v_x - v_{x_0}); \qquad \text{(V)}\quad \int_{t_0}^{t} P_t\,dt = m_1(v - v_0).$$

Work equations:

$$\text{(VI)}\quad \int_{x_0}^{x} P_x\,dx = m_1\frac{v_x^2 - v_{x_0}^2}{2}; \qquad \text{(VII)}\quad \int_{s_0}^{s} P_t\,ds = m_1\frac{v^2 - v_0^2}{2}.$$

$$\text{(VIII)}\quad \int_{s_0}^{s} P_t\,ds = \int_{u_0}^{u} R\,du = \int_{x_0}^{x} P_x\,dx + \int_{y_0}^{y} P_y\,dy + \int_{z_0}^{z} P_z\,dz.$$

The symbols agree with our previous usage: P_x is typical for any fixed line X; P_t applies to the direction of the simultaneous tangents to the paths; P_n and R are the totals of normal and resultant force. When P_x is of constant magnitude, equations (IV) and (VI) become

$$\text{(IX)}\quad P_x(t - t_0) = m_1(v_x - v_{x_0}); \qquad \text{(X)}\quad P_x(x - x_0) = m_1\frac{v_x^2 - v_{x_0}^2}{2}.$$

When P_t is of constant magnitude,

$$\text{(XI)}\quad P_t(t - t_0) = m_1(v - v_0); \qquad \text{(XII)}\quad P_t(s - s_0) = m_1\frac{v^2 - v_0^2}{2}.$$

Note that the constancy of P_t does not carry with it the constancy of the corresponding P_x, P_y, P_z, R. The other integrals occurring in equation (VIII) may remain, even in cases where the first integral can be written as a product.

68. In entering upon the consideration of pure rotation (rotation about a fixed axis), let us first take up the work equation in the form that applies to tangential force. The entire work at every part throughout the body is comprehended in the equation (§ 58),

$$\int_0^{m_1}\int_{s_0}^{s} dP_t\, ds = \int_0^{m_1} dm\,\frac{v^2 - v_0^2}{2}. \tag{1}$$

It is no obstacle to forming the indicated sum that the elements dP_t have various directions. We are dealing with work which is a scalar quantity; and this feature removes the limitation that the group of forces must stand in any particular relation of direction, either at the same instant or at successive instants. For pure rotation it is, however, necessary that the angular displacement $d\gamma$ and the angular velocity ω be common to all parts of the mass at the same time. Consequently $ds = r\,d\gamma$, and $v = r\omega$, at the (constant) distance r of any differential mass from the axis; and further $\int_0^{m_1} r^2\,dm = I_o$ is constant (§ 148). If now $r\,dP_t$ be denoted by dM_t, and $\int_0^{m_1} r\,dP_t$ by M_t, equation (1) takes the form

$$\int_{\gamma_0}^{\gamma} M_t\,d\gamma = I_o\frac{\omega^2 - \omega_0^2}{2}. \tag{2}$$

The quantity dM_t is termed the **moment** of the tangential force dP_t; and M_t is the **total moment** for the complete group of such forces. The total moment of tangential force, then, is so determined that its integral with respect to angular displacement round the axis is the entire work during a pure rotation, summed for the whole body. In order to preserve consistency in the signs, dM_t must be considered positive for forces whose work is positive during positive angular displacement (§§ 10, 56); the total moment M_t is then the

algebraic sum of its elements. The constituents of the work integral in equation (2) will be positive so long as M_t and $d\gamma$ have the same sign; negative when those factors have opposite signs. Moment may of course be constant or variable; if M_t is constant,

$$M_t(\gamma - \gamma_0) = I_o \frac{\omega^2 - \omega_0^2}{2}. \tag{3}$$

69. At any differential mass the entire work is that of the tangential force; whatever work is contributed by any forces applied there must be due to their projections upon the tangent. Let such projection of any force dP be dP_2; then its contribution to work is $r\,dP_2\,d\gamma$, and its moment is $r\,dP_2$, according to our previous definition. But in all cases of pure rotation, all tangents are in planes perpendicular to the axis, and the projection of dP can be carried out in two stages. These are: (1) Projection into a plane perpendicular to the rotation-axis (the component parallel to the axis does no work); call the result dP_1; (2) Projection of dP_1 upon the tangent (the radial component does no work). Then the moment of dP may be written also $h_1 dP_1$, if h_1 is the distance from the axis to the line of action of dP_1; for $r \cos\beta = h_1$, if $dP_2 = dP_1 \cos\beta$, because r is perpendicular to dP_2. The following rule applies, therefore, in expressing the moment of any element of force: First, project the force into a plane perpendicular to the rotation-axis; and second, multiply the component thus obtained by the distance of its line of action from the axis. The length which is the second factor is often called the **lever-arm** of the (projected) force. Hence a force may be taken as acting upon any differential mass through which its prolonged line of action passes, without affecting its moment. To this extent the statement is true, "A force may be considered as acting anywhere in its own line."

70. If Z be assumed as the rotation-axis, dP_z is everywhere zero, and the work of the tangential force can be written (§ 58),

$$\int_0^{m_1}\int_{s_0}^{s} dP_t\, ds = \int_0^{m_1}\int_{x_0}^{x} dP_x\, dx + \int_0^{m_1}\int_{y_0}^{y} dP_y\, dy. \qquad (1)$$

For positive angular displacement $dx = -y\,d\gamma$, and $dy = x\,d\gamma$, besides $ds = r\,d\gamma$. If these values be substituted in (1), we obtain

$$\int_0^{m_1}\int_{\gamma_0}^{\gamma} r\,dP_t\, d\gamma = -\int_0^{m}\int_{\gamma_0}^{\gamma} y\,dP_x\, d\gamma + \int_0^{m_1}\int_{\gamma_0}^{\gamma} x\,dP_y\, d\gamma. \quad (2)$$

But $-y\,dP_x$ and $x\,dP_y$ are the moments respectively of dP_x and dP_y. Denoting then by dM_x and dM_y, we have

$$-\int_0^{m_1} y\,dP_x = M_x, \text{ and } \int_0^{m_1} x\,dP_y = M_y.$$

From equation (2), therefore, we may conclude

$$M_t = M_x + M_y. \qquad (3)$$

By like reasoning the tangential and resultant forces can be shown to have equal moments, or $M_t = M_R$. And the principle can be announced for pure rotation that the moment of all the resultant or tangential forces is the algebraic sum of the total moments of any pair of their rectangular components. The limits of this proposition can be extended, but it serves all present purposes in the above form.

71. When two forces constitute a couple, the algebraic sum of their moments is the **moment of the couple**. This moment will depend upon the direction of the axis about which rotation is supposed to take place. Consider it first with respect to any axis normal to the plane (called the **plane of the couple**) which contains the parallel lines in which the forces act. Let the forces (equal in magnitude) be distinguished as dP' and

dP''; and let their lever-arms be h' and h''. Then the force-moments separately are $h'dP'$ and $h''dP''$. Since the forces are contrary in direction, the moment of the couple is the numerical difference of the separate moments if h' and h'' are in the same direction; and it is the numerical sum if they are oppositely directed, when measured outward from the axis in both cases. In either event, the algebraic sum of the moments is $h\,dP$, where h (the arm) is the distance between the forces of the couple, and dP denotes their common magnitude. Consequently this moment is independent of the position of the axis, provided it be normal to the plane of the couple. On account of this equal relation as regards moment, any normal to its plane is called the **axis of the couple.** If we observe the limitation named, the relative situation of axis and couple is without influence on the couple-moment. Hence, if any particular normal axis is given, and it is allowable to apply the forces differently in their plane, but always as a couple, the moment will preserve continuously a constant value, if the forces be shifted in position and varied in direction without leaving their original plane, when the arm shifts and turns with them, as if in rigid connection. Even the magnitude of the forces and the length of the arm may be altered, subject to the condition that the product of the two remains constant.

72. The moment of a couple is next to be expressed when the rotation-axis and the couple-axis are inclined at any angle α. Think of the couple as shifted in its own plane until the line of one force intersects the rotation-axis and is perpendicular to it. Then the moment of that force becomes zero; the moment of the couple is that of the other force, and is evidently $h\,dP\cos\alpha$. It is a matter of simple geometry to prove that this moment is again the same for all parallel axes,

and consequently independent of such changes in the elements of the couple as have been discussed in the preceding case.

Consistently with these results, the moment $h\,dP$ may be regarded as a vector magnitude, whose direction is normal to the plane of the couple, and whose projection upon any rotation-axis gives the couple-moment with respect to that axis; the "direction of a couple" is that of its moment-vector. A complete graphical representation of a couple may be made, if a line of proper direction is drawn to scale and shows the moment, while sign harmonizes with conventions previously established. In order to cover the last point, the forward direction of the vector, and the turning that the (unhindered) action of the couple would produce, should be in the same relation as the advance and the turning of a right-handed screw (cf. § 10, (3)). Then the moment of the couple and the angular displacement producible by it will be positive or negative together, and the proper sign will result for the work of the forces.

The moment of any force can be included in such a conception of vector magnitude. The vector being associated with and laid off upon the rotation-axis for which the moment is expressed, it has direction for a couple, but no position, because couple-moment has the same value for the entire group of parallel axes. When we are dealing with a single force, however, whose moment varies in magnitude according to the axis chosen, the vector is individualized in the group, and the length representing it must be laid off upon that particular axis. In forming the sum represented by $\int_0^{m_1} r\,dP_t$ (§ 68), there is from this point of view an algebraic addition of vector quantity in the same line; viz. the rotation-axis. The connection of moment with an axis will be symbolized thus: $M(x)$ — moment of force about the axis X, or moment of a couple about an axis parallel to X.

Let the force $\widehat{dP}$ act upon an element of mass distant $\widehat{r}$ from a point O, α being the angle between these vectors. The vector part in the product $\widehat{r} \cdot \widehat{dP}$ represents the moment of dP about an axis through O, normal to the plane of the vectors. For the magnitude of the product is $r\,dP \sin\alpha$; its line is normal to the plane referred to; and conventionally its direction agrees with the specification for the graphical representation of moment. The moment about any other axis through O is obtainable by projection of the above vector magnitude.

Work and moment are both products of force and length. But the former is scalar; the force and the length are measured in the same line. The latter is a vector; the force and the length are perpendicular to each other. The unit in each case involves the dyne and the centimeter; but a distinction which is not too artificial may be made in the order of the factors. The erg is the dyne-centimeter. The unit moment is rather the centimeter-dyne.

73. The idea of force-moment has been first presented in connection with pure rotation, when work is being considered. But the scope of the conception is wider than this one relation; the same quantity enters as an essential factor into other important expressions. When pure rotation occurs, the tangential force for a differential mass at any distance r from the axis is $dP_t = dm\, r\frac{d\omega}{dt}$; its moment is $r\,dP_t = dm\, r^2\frac{d\omega}{dt}$; and consequently the total moment of the tangential force

$$M_t = I_o\frac{d\omega}{dt}. \tag{1}$$

Equation (2), § 68, is the integral of the equation above with regard to γ. But it may also be integrated with respect to time, giving

$$\int_{t_0}^{t} M_t\,dt = I_o\,(\omega - \omega_0). \tag{2}$$

The name to be applied to the quantity in the first member is plainly suggested by the conformity in its type with impulse of force; it is **impulse of** (tangential) **force-moment.** The char-

acteristic term $I_0\,\omega$ of the second member being analyzed as $\int_0^{m_1} r\,[dm\,r\omega]$, it can be described as the **total moment of** (the resultant) **momentum.** In both members the signs are determined by considerations that have been dwelt upon already. Equation (2) may then be read: The impulse of tangential force-moment is measured by the change in the total moment of (resultant or tangential) momentum produced during the interval. The element of momentum is here treated as a vector quantity having position, like force, and the moment is formed for both force and momentum by the same rules so far as magnitude is concerned; the sign of moment of momentum is that of the existing angular velocity — not acceleration.

The first member of equation (2) can be written also in the form (r being constant) $\int_0^{m_1} r \int_{t_0}^{t} dP_t\,dt$, and is then described appropriately as the **total moment of** (tangential) **impulse.** This analysis is instructively parallel with that of the second member, for

$$\int_{t_0}^{t} dP_t\,dt = dm\,(v - v_0) = dm\,r\,(\omega - \omega_0). \qquad (3)$$

74. The general outcome of §§ 68–73 is to establish equations particularly adapted to the circumstances of pure rotation, in that *linear* displacement, velocity, and acceleration are replaced by the corresponding *angular* quantities. The fundamental constraint in a rigid body expresses itself for rotation as a common value in all parts for angular displacement, velocity, and acceleration. Compare § 73, (1) with § 67, (II); § 73, (2) with § 67, (V); § 68, (2) with § 67, (VII). On making the comparison indicated between parallel equations for translation and rotation it must be noticed in addition how essential the element of position becomes in the latter case. Not the amount of force, but the force-moment, is decisive. The conception of moment of inertia contains not only the idea

of mass-quantity, but also that of mass-distribution. Finally, we deal with the moment of both impulse and momentum. The importance of all these elements, as regards influence upon the forms of result here in question, depends upon their distance from the axis; and the meaning to be attached to the word "moment" in this connection is exactly "importance." Further, it may be noted, in passing, that elements of force in couples, and elements of momentum in pairs, which disappear by cancellation from equations of motion and impulse, are preserved in the "moment-equations." The moments of force and momentum are conceptions which aid in securing a complete account of the dynamical phenomena (cf. § 64).

In pure rotation the circular path of each part of the body is predetermined by the constraints; the speed in the circle is the element that can be varied. Hence follows the prominence of tangential forces and their total moment for these conditions. The moments of stresses occur necessarily in self-cancelling pairs; they cannot enter effectively into the total. Internal forces being always stresses, the effective moment must be due to external forces; and from these we may exclude as a second group forces whose directions intersect the rotation-axis.

Stresses then affect neither the translation nor the rotation of a rigid body; and in view of the fundamental condition of rigidity, their work must always be zero. In the supposed cases they are without influence upon the phenomena of motion.

Wherever the total moment M_t is always zero, the angular velocity of a rigid body rotating about a fixed axis must remain constant, in conformity with equation (2), § 68. Experience shows the requisite constancy of angular velocity so long as the moment is that of internal force alone, and from this side confirms the consequences of the postulate in § 47. Observe particularly that the total moment would not be zero,

though the internal forces were equal and contrary, unless both elements in each pair acted in the same line, and thus constituted a stress.

75. Neither the kinetic energy nor the moment of momentum of a body vanishes when it is rotating about an axis that passes through its centre of mass, because the moment of inertia for such an axis is not zero. It has already been shown (§ 62) that the centre of mass is not the centre for kinetic energy. For each rigid body in its relation to a particular rotation-axis, however, there is always an average distance such that, if the conditions as regards resultant velocity prevailing there were common to the entire mass, the total kinetic energy would retain its actual value. A point at that distance from the axis is a centre for kinetic energy in the same sense that the centre of mass is a centre for momentum. The condition referred to is satisfied by a point at the distance k_o from the rotation-axis determined by the relation

$$m_1\frac{k_o^2\omega^2}{2} = I_o\frac{\omega^2}{2}; \quad \text{or} \quad k_o^2 = \frac{I_o}{m_1}. \tag{1}$$

The same point is a centre for moment of momentum also; for the average distance from the axis here would be fixed by the condition

$$k_1[m_1k_1\omega] = I_o\omega; \quad \text{or} \quad k_1^2 = \frac{I_o}{m_1} = k_o^2. \tag{2}$$

The length k_o that fulfils equations (1) and (2) is called the **radius of gyration** for the particular body and axis.

The same thought is expressed by saying: Kinetic energy and moment of momentum would not be changed when a rigid body is in pure rotation if the entire mass could be condensed into a point at the outer extremity of the radius of gyration, the angular velocity being unaltered.

76. In problems of physics, and in technical applications of Mechanics, it is often required to express the work of forces in relation to the time during which they are active, and to compare the rapidity with which given effects can be produced that are measured directly or indirectly as work. For such purposes the time-rate of work $\left(\frac{dW}{dt}\right)$ needs to be considered; and the word **power** has been appropriated to denote a quantity proportional to that coefficient. The unit for numerical specification of power that is most closely adapted to the standards thus far fixed would be one erg per second. A larger unit, however, proves more convenient for the range of the chief applications; and a power-unit defined as $10^7 \frac{\text{ergs}}{\text{sec.}}$ is in common use under the name of **watt**. A still larger unit is the **horse-power**, standardized at 746 watts.

Since work may be positive or negative, so may its time-rate; but power is usually thought of from one side only of the transaction, and construed as a positive magnitude. Contrivances by means of which positive work is done (for example, motors) are said to supply or furnish power; those in which resistances predominate, whose work is negative (for example, machines), are said to absorb power when they are in operation.

The quantitative expressions for power can be derived from those for work; this shall be done for translation first. The work-element dw of tangential force being $dP_t\, ds$, the power is $\frac{dw}{dt} = dP_t \frac{ds}{dt} = dP_t v$; and (I) $\frac{dW}{dt} = P_t v$. Similarly, for force parallel to X, $\frac{dw}{dt} = dP_x \frac{dx}{dt} = dP_x v_x$; and (II) $\frac{dW}{dt} = P_x v_x$. For pure rotation, $dw = dM_t\, d\gamma$, $\frac{dw}{dt} = dM_t \frac{d\gamma}{dt} = dM_t \omega$; and (III) $\frac{dW}{dt} = M_t \omega$.

The above equations, being based upon instantaneous values, are valid for either constant or variable force and force-moment. If these quantities are constant, the corresponding speed-factors must be constant as well, in order that the power should be constant. But the speed cannot be constant, when the total tangential force or its total moment is included, unless these latter quantities are zero (§ 67, (VII), § 68, (2)). In other words, for "steady motion" (constant speed) the power must on the whole be zero, the power supplied just balancing that which is absorbed. And if we speak of power in such a case (as we habitually do), we refer to the positive work of the forces, and the power furnished, as the prominent ideas.

77. It has been shown earlier (§ 20) that uniplanar motion of a rigid body can be regarded kinematically as a combination of translation with rotation. The velocity of the translation may be that of any point rigidly attached to the body; but the rotation-axis must then pass through the point selected. The angular velocity of the rotation at a given instant is the same, however, for any such axis. When a body is rigid its centre of mass is effectively in rigid connection with it (§ 146); so that uniplanar motion can be treated as a combination of translation parallel to the guide-plane with the velocity of the centre of mass, and rotation about an axis through the same point normal to the guide-plane. The admissible kinematical combinations are indefinite in number; but the choice of the centre of mass gives a statement which is unique, in that the dynamical equations apply independently to the two partial motions (§§ 63, 65, 66). For the translation we can use the equations of motion, impulse, and work, § 67, (I) to (VIII). The motion relative to the centre of mass is here rotation, and it will proceed according to equations of the types § 68, (2), § 73, (1), (2). The force-moments of those equations are to be taken

about an axis normal to the guide-plane, and passing through the centre of mass (§ 63, (1)). If that axis is parallel to Z, the moments will be those of external forces only, parallel to X and Y (§§ 74, 70, (3)).

The presentation of forces relative to the centre of mass as grouped in couples has been carried out in § 65; and by adopting a simple device we may explain, for the conditions here under discussion, the appearance of such couples as an accompaniment, when external forces are transferred to the centre of mass (§ 61, (I)). Let an external force dP act at any differential mass, and suppose a parallel stress $\pm dP$ applied at the centre of mass of the body. The actual motion will not be changed thereby, for, as we have seen, stresses have no influence upon either translation or rotation. The translation *does* proceed as though produced by a group of forces whose type is $+dP$ of the supposed stress; acceleration, impulse, and work all being of the requisite values. These elements of force, moreover, cannot affect rotation about any axis through the centre of mass. There remains a couple composed of dP (actually applied), and $-dP$ of the stress, whose moment is the actual moment of dP for any axis passing through the centre of mass. Such couples cannot affect translation; and the rotation about the axis normal to the guide-plane must proceed as though produced by couples of this type. This point of view regarding uniplanar motion is summarized in the proposition: The elements of the rotation relative to the axis through the centre of mass of the body are determined just as if that axis were fixed, and the external forces acted unchanged in magnitude, direction, and position.

Some care must be exercised to see the real meaning of these statements, for those external forces which appear in both groups of equations have apparently a double function in producing both translation and rotation, while any force is of course fully accounted for by one measure

of its effect. The question can be raised about the effective or "unbalanced" elements of external force only, because couples affect rotation alone, and do not appear twice.

If a fixed axis passes through the centre of mass, and a force of amount dP be applied in a plane perpendicular to the axis, with moment of amount dM, the axis must necessarily supply a reaction of amount $-dP$ in order that the prescribed motion may occur, leaving the centre of mass at rest. The force dP applied and the reaction $-dP$ form a couple. If, now, the force dP be active as before, but the axis be relieved from the constraint of remaining fixed, and allowed to move into parallel positions, the departure from rest of the centre of mass will measure the extent to which the axis fails to supply $-dP$. If no reaction be exerted (*i.e.* if dP acts alone) the acceleration of the centre of mass will take place to the full extent $\frac{d^2\bar{x}}{dt^2} = \frac{dP}{m_1}$. And the consequent displacement will accompany and modify that due to the rotation. The actual displacement will be composed of the two elements by a sort of addition or subtraction, according to situation in the body. The parts where an external force is applied get ahead by receiving a larger share proportionately in the automatic distribution of force through a rigid body. Referring now to the previous device of applying a stress $\pm dP$, the $+dP$, by neutralizing $-dP$ (the reaction), is the equivalent of relaxing the constraint that would hold the axis fixed.

78. When the body is rigid and the motion is uniplanar the work equation, § 63, (2), may be written to include the complete change of energy for a given interval in the form

$$\int_{\bar{s}_0}^{\bar{s}} P_t d\bar{s} + \int_{\gamma_0}^{\gamma} M_t d\gamma = m_1 \frac{\bar{v}^2 - \bar{v}_0^2}{2} + I_c \frac{\omega^2 - \omega_0^2}{2}. \qquad (1)$$

P_t as here used must be taken to mean the sum obtained by projecting all external forces upon the tangent to the path of the centre of mass; and $\bar{s}$ is a coördinate measured in that path. I_c is calculated for an axis normal to the guide-plane at the centre of mass; and M_t is the total moment of external forces for that axis. At any position the kinetic energy

$$E = m_1 \frac{\bar{v}^2}{2} + I_c \frac{\omega^2}{2}. \qquad (2)$$

Let I_t be the moment of inertia of the body for the instantaneous axis (§ 21); and $\bar{r}$ the distance of that line from the centre of mass. Then (§ 150)

$$E = \frac{\omega^2}{2}(m_1\bar{r}^2 + I_c) = \frac{\omega^2}{2} I_t. \tag{3}$$

The kinetic energy is given with reference to the instantaneous axis by an expression of the same type as though the axis were permanent; but with the important difference that I_t is not in general constant. The constancy of this quantity is not in itself a sufficient condition, however, though a necessary one, for securing pure rotation as a particular case of uniplanar motion. The locus of equal values for moment of inertia among admissible axes is a cylindrical surface with the centre of mass on its axis. If $\bar{r}$ is constant, the instantaneous axis is confined to this surface, but not to one generator of it.

79. It is an obvious consequence of the preceding discussions that the systems of external force applied to a rigid body can be varied in many ways, while the equations of motion, impulse, and work, remain formally unchanged. In fact, this idea has already been used in a rudimentary way for purposes of analysis and description. Confining ourselves to uniplanar motion, the effects in translation will be indistinguishable one from another, provided the total force in every direction is the same. Stresses and couples may be added or removed; forces may be shifted in position, concentrated upon smaller areas, or distributed over larger surfaces; all without modifying this part of the result. The effects in rotation about the axis through the centre of mass will present a given series of values under any conditions that provide the necessary force-moments. For example, it is a matter of indifference for the rotation what forces are added or dropped, whose lines of action intersect the axis. It seems sufficiently evident that such

changes in the forces and their arrangement can be adjusted to compensate each other on the whole, and leave the phenomena of the motion entirely unaffected. **Force-groups** are said to be **equivalent** in their action upon a body within a given interval, if the quantities of force, impulse, and work, are equal when calculated from the fundamental equations of § 60, proper account being taken of direction as well as magnitude. The total force and the force-moment about the axis through the centre of mass are the essential elements in fixing these quantities; hence, they are important as furnishing a criterion of equivalence when there is question of substituting one force-group for another. These two elements are included in the conception of **resultant**, which is determined for uniplanar motion by specifying the total force parallel to the guide-plane, and the total moment about a normal axis at the centre of mass. All equivalent force-groups have equal resultants.

Let XY be the guide-plane; then the rotation-axis is parallel to Z at the centre of mass, and $dP_z = 0$ everywhere in the body. Let R be a force of such magnitude and direction that

$$R^2 = P_x^2 + P_y^2 \quad (1); \qquad tg\,\alpha = \frac{P_y}{P_x}, \qquad (2)$$

the angle α being measured from X to R. Then the projection of R upon any line will show the actual total force parallel to that line. Further, let the position of R be so chosen that

$$h_1 R = M_t = M_x + M_y, \qquad (3)$$

all moments being taken about the axis at the centre of mass. Then R is the resultant of the force-group as given, according to the definition of this section.

80. A second method that is frequently employed in specifying the resultant separates it into a force and a couple. Think of R, determined by equations (1) and (2), § 79, as acting at

the centre of mass. It will meet the requirements of the translation in all respects — force, impulse, and work. In addition, let a couple be assumed, whose axis is parallel to Z, and whose forces P and arm h have any values that satisfy the relation $hP = M_t$. The couple is adequate to maintain the actual rotation as regards angular acceleration and velocity, and work. An advantage of this method is that the equivalent system is broken up into two independent parts, in adjustment with the standard analysis of the motion (§ 77). The connection between the two modes of statement is found in the fact that R, acting "off centre" by the length h_1, is the equivalent of R at the centre of mass and the couple-moment $h_1R = hP = M_t$.

When we are dealing with pure rotation, the important moment is usually that calculated for the actual rotation-axis, which we will now call M; whereas M_t has been calculated for a parallel axis at the centre of mass. As a consequence of the idea of resultant, the relation between the two moments is $M = \bar{r}R_t + M_t$; $\bar{r}$ being the distance of the centre of mass from the axis, and R_t the tangential component of R. Compare equation (1), § 78, which becomes for the special suppositions

$$\int_{\gamma_0}^{\gamma} (\bar{r}R_t + M_t)\, d\gamma = (m_1\bar{r}^2 + I_c)\frac{\omega^2 - \omega_0^2}{2} = I_o\frac{\omega^2 - \omega_0^2}{2}. \quad (1)$$

It follows of course from (1) that the same angular acceleration results by calculating for either axis. Since $R_t = m_1\bar{r}\dfrac{d\omega}{dt}$, we have

$$I_o\frac{d\omega}{dt} = \bar{r}\left(m_1\bar{r}\frac{d\omega}{dt}\right) + M_t; \quad \frac{d\omega}{dt} = \frac{M_t}{I_o - m_1\bar{r}^2} = \frac{M_t}{I_c}. \quad (2)$$

In the discussion of problems, an external force of finite magnitude is often spoken of as acting along a line (in the strict geometrical sense), while from the standpoint of physical possibility a finite force requires a finite area for its application, such as the cross-section of a rod or wire or rope. The supposed concentration upon a line is to be viewed as a substitution of its resultant for a (partial) group of forces, that are in

reality distributed over such a finite area. With this interpretation we are at liberty to consider a finite force as acting in a line. A like use of an ideal equivalent system with finite mass at a point underlies the second statement of (I), § 61, and the conception of radius of gyration (§ 75). If we imagine a concentration of both force and mass, the equation $M_t = I_0 \frac{d\omega}{dt}$ is susceptible of interpretation as the equation of motion for an ideal equivalent system at unit distance from the rotation-axis. Suppose tangential force equal to M_t to act upon mass equal to I_0; it will produce (linear) acceleration equal to $\frac{d\omega}{dt}$. It is numerical equality in all three cases, the suppressed factors being unity.

81. Attention has been directed thus far to certain quantitative features of translation, rotation, and their combination in uniplanar motion. While engaged with that aspect of the matter we have taken for granted that the motion was always of a given type, without making particular inquiry about the conditions necessary to preserve its character. Such conditions can be realized most comprehensively by means of mechanical constructions like guides, cranks, linkages, which impose constraints by the limitation of selected points and surfaces to definite loci. This is, indeed, only one method of supplying force that is requisite, and which it is possible to apply in other ways. The distinctive advantage of the method is that forces of any magnitude needed to guide the motion and make the constraints effective are furnished, up to the point where the yielding of the material used ceases to be negligible. We shall use the word **constraints** in agreement with this idea; they are forces that control the type of the motion, but not its rapidity; they aid in fixing the paths, but not the speeds in them. The constraints supplement the other forces automatically with exactly those elements which bring the completed force-group into adjustment with a prescribed type of motion. For attaining the adjustment we may rely upon constraints alone; or upon regulation and disposi-

tion of other forces; or upon combinations of constraints with other forces. Otherwise expressed, the constraints may be more or less nearly complete, leaving different ranges of possibility for actual motion. Where a requirement is to be met by the joint action of constraints and other forces, the extent to which the former must be drawn upon can be calculated when the latter are definitely given. The ideas of equivalent system and resultant open the way to formulate briefly the essentials for each typical case. The consideration of the three types here to be included will be taken up in turn. Beginning is made with pure rotation because it furnishes important clews to the conditions for either translation or uniplanar motion.

In the ordinary form of steam-engine the piston-rod is guided completely, and restricted to right-line translation; the crank is compelled to rotate about a fixed axis; and the pitman is confined to uniplanar motion. Whatever forces are otherwise applied are so modified by the constraints that the above motions, and those only, can occur while the mechanism preserves its integrity. A garden-roller has uniplanar motion while rolling straight ahead, the guide-plane being parallel to the circular section. But the constraint of the surface rolled is only partial; the roller may swerve as it moves forward, and depart from uniplanar motion by introducing rotation about an axis parallel to the guide-plane. In order to secure the desired result, with such incomplete constraints, the other forces applied must fulfil certain conditions.

82. Let a given rigid body be in rotation with angular velocity ω about Z, its centre of mass being contained in the XY plane at a known distance, $\bar{r}$, from the axis. The forces parallel to the reference-axes acting on any differential mass at (x, y, z) are

$$dP_x = -\frac{d\omega}{dt} y\, dm - \omega^2 x\, dm; \qquad (1)$$

$$dP_y = \frac{d\omega}{dt} x\, dm - \omega^2 y\, dm; \qquad (2)$$

$$dP_z = 0. \qquad (3)$$

The totals of these forces are

$$P_x = -\frac{d\omega}{dt}\bar{y}m_1 - \omega^2\bar{x}m_1; \tag{4}$$

$$P_y = \frac{d\omega}{dt}\bar{x}m_1 - \omega^2\bar{y}m_1; \tag{5}$$

$$P_z = 0. \tag{6}$$

The total moments about the reference-axes are found to be

$$M(x) = -\int_0^{m_1} z\,dP_y = -\frac{d\omega}{dt}\int_0^{m_1} xz\,dm + \omega^2\int_0^{m_1} yz\,dm; \tag{7}$$

$$M(y) = \int_0^{m_1} z\,dP_x = -\frac{d\omega}{dt}\int_0^{m_1} yz\,dm - \omega^2\int_0^{m_1} xz\,dm; \tag{8}$$

$$M(z) = \int_0^{m_1} x\,dP_y - \int_0^{m_1} y\,dP_x = \frac{d\omega}{dt}\int_0^{m_1}(x^2+y^2)\,dm = \frac{d\omega}{dt}I_z. \tag{9}$$

Let now the position be so chosen that the centre of mass is on X; the essentials of this situation will be repeated continually with reference to new axes in the same relation to the body, if X is always the radius-vector drawn to the centre of mass. The analysis will be followed that supposes a resultant force active at the centre of mass, and accompanied by couples. Let the special values of forces and couple-moments for the position $(\bar{x} = \bar{r}, \bar{y} = 0, \bar{z} = 0)$ be P_1, P_2, P_3, M_1, M_2, M_3. Then

$$P_1 = -\omega^2\bar{r}m_1; \quad P_2 = \frac{d\omega}{dt}\bar{r}m_1; \quad P_3 = 0.$$

In determining M_1, M_2, let it be noted, first, that the moments for axes parallel to X and Y at the centre of mass are the same as for those reference-axes, because $\bar{z} = 0$; and, secondly, that the integrals in equations (7) and (8) now assume definite values which are retained for the similar axes in other azimuths round Z. Finally we have $M(z) = \bar{r}P_2 + M_3$ (§ 80).

The rotation is determined in magnitude according to equation (9); and we shall regard $M(z)$ as given; for the lines of the constraints, with which we are at present especially concerned, intersect the rotation-axis, and contribute nothing to the moment in question. Eliminating that element, we have to consider the force of the resultant,

$$\widehat{R} = \widehat{P}_1 + \widehat{P}_2; \tag{10}$$

and its couple,

$$\widehat{M} = \widehat{M}_1 + \widehat{M}_2; \tag{11}$$

the process of composition, when the axes intersect, being the legitimate reversal of the projective property proved in § 72. Let the components of other forces parallel to the reference-axes have given totals X_1, Y_1, Z_1; and the totals for the supplementary constraints be X_a, Y_a, Z_a. The conditions must be fulfilled,

$$X_1 + X_a = P_1; \quad Y_1 + Y_a = P_2; \quad Z_1 + Z_a = P_3 = 0. \tag{12}$$

Distinguish similarly between $M(x)_1$, $M(y)_1$; and $M(x)_a$, $M(y)_a$; then it is further necessary that

$$M(x)_1 + M(x)_a = M_1; \quad M(y)_1 + M(y)_a = M_2. \tag{13}$$

Therefore, in order that the supposed pure rotation about Z may continue with unchanged type, equations (12) and (13) must be satisfied constantly in relation to the changed planes XZ and YZ. The angular acceleration and velocity are necessary data for calculating P_1, P_2, M_1, M_2, and obtaining definite values for the constraints required when the other forces and moments are given. Equation (9) determines $\frac{d\omega}{dt}$ for given $M(z)$; and its integral with respect to γ fixes ω. But $\frac{d\omega}{dt}$ is not of itself sufficient to determine X_1 and Y_1; that acceleration depends upon their joint moment, not upon the forces.

Such couples as M_1, M_2, have no effect in producing angular velocity about their own axes. We shall call them **directive couples**; they secure the constant direction of the rotation-axis. Their function, as they are exerted upon the body in the planes parallel to YZ and XZ, is to maintain it in the necessary relation to the same line as rotation-axis. But notice that they *produce* a moment requisite under the circumstances; they do not *neutralize* one that already exists (cf. § 137). The required elements of force in their actual distribution through the mass do have such positions that these moments about X and Y result. Therefore the equivalent system of (external) forces must include them also.

83. Even for the comparatively simple instance selected, the discussion has shown a proper regulation of the forces and couples to be a complicated matter. If the resultant be expressed as a single force, the same difficulties reappear in the necessary changes of its magnitude, direction, and position. So the practical advantage is clearly apparent, of relying upon constraints and their automatic adjustments to the necessities. At the same time the general endeavor is so to design actual constructions as to minimize the constraints needed. This is done in order to avoid friction, as well as the jar and vibration attendant upon the action of forces and couples with varying directions and magnitudes. It is therefore profitable to consider some general possibilities connected with the results just obtained.

I. The couples M_1 and M_2 vanish when

$$\int_0^{m_1} xz\,dm = 0; \quad \int_0^{m_1} yz\,dm = 0.$$

By mere inspection this is seen to result for certain forms of symmetry in the body; the condition is always fulfilled if Z is a principal axis (§ 151), for the origin.

II. If the motion is steady (ω constant), $\dfrac{d\omega}{dt}=0$, $P_2=0$; and even though the conditions $M_1=0$, $M_2=0$, are not fulfilled, the products containing $\dfrac{d\omega}{dt}$ disappear from those moments. If the axis is a principal axis for the origin (the motion being steady), the couple of the resultant disappears, and the latter reduces to a radial pull through the centre of mass inward toward the axis, and of amount $\omega^2\bar{r}m_1$.

III. If the centre of mass lies on the rotation-axis ($\bar{r}=0$), the force $\widehat{R}$ of the resultant vanishes, whether the motion be steady or not. For variable ω, couples only are required; if the other forces do not constitute such couples, the constraints must complete the couple-system. If, then, the centre of mass is situated upon a rotation-axis, which is a principal axis for that point, and the (other) external forces reduce to a couple whose axis is parallel to the rotation-axis, no constraints are necessary. Such a rotation-axis is called a **free axis**.

When $\bar{r}=0$ the location of the XZ plane becomes indeterminate. Any plane through Z may be assumed, provided it be continuously changed so as to retain the same relation to the body.

All practical cases can be covered by combining the ideas of the above headings. Enough has been said to make apparent the advantage of having rotating bodies well "balanced" ($\bar{r}=0$, and axis a principal axis for the centre of mass) by means of counterpoises or other features of design.

84. Pure rotation is that form of uniplanar motion which results from a particular adjustment that causes the centre of mass to move in a circular path (cf. § 78). This has for consequence a special value of $\widehat{R}$ (§ 82, (10)), which need not obtain for less narrow suppositions. Take the guide-plane as XY, and let it contain the path of the centre of mass. Then

P_z must be zero; but P_x and P_y may have any values without breaking away from uniplanar motion. The translation will proceed in accordance with the equations,

$$P_x = m_1 \frac{d^2\bar{x}}{dt^2}; \quad P_y = m_1 \frac{d^2\bar{y}}{dt^2}; \quad P_z = 0. \tag{1}$$

The rotation-axis must remain parallel to Z; and in order to fulfil this condition directive couples will be requisite, unless the rotation-axis is a principal axis for the centre of mass. We shall continue to denote the directive couple-moments by M_1 and M_2; they may be calculated independently of the reference-axes by means of which the translation is described. Let their axes X' and Y' be any two lines at right angles and parallel to the plane XY; but changed as in § 82 so as to preserve the same relation to the body while it turns, in order that the characteristic integrals in the expressions for M_1 and M_2 may have constant values.

The magnitude of the rotation will be governed by the equation

$$M_3 = I_c \frac{d\omega}{dt}, \tag{2}$$

if M_3 is the total moment of all external forces about the rotation-axis through the centre of mass. The equations to be satisfied by the constraints are with parallel notation to § 82, (12), (13),

$$Z_1 + Z_a = 0; \quad M(x')_1 + M(x')_a = M_1; \quad M(y')_1 + M(y')_a = M_2. \tag{3}$$

85. The idea has already been insisted upon repeatedly, that external forces applied to a rigid body are equivalent to the same forces transferred to the centre of mass, together with couples. The translation is guided by the force-term of the resultant, while the effects of couples are shown during motion relative to the centre of mass. The couple of the

resultant can be stated in terms of its components arising by projection upon the reference-axes; and their effects have been found, either in developing angular velocity about their own axes, or in steadying relatively to a rotation-axis. The latter form of action becomes necessary only when angular velocity or acceleration due to another (component) couple exists. If no acceleration relative to the centre of mass is allowable, both types of couple must be excluded. The characteristic of translation, stated as equal velocities and acceleration everywhere, or as constancy of direction for every line, is incompatible with acceleration relative to the centre of mass. And the resultant must therefore reduce to a force applied at that point, the component couple-moments being zero for every axis. On account of the projection relation, the couple is zero for all axes if the components for the three reference-axes are zero.

The equations of condition for the persistence of translation are

$$\widehat{R} \text{ (any value); and } \widehat{M} = 0.$$

If the forces otherwise applied yield a couple-moment about any axis, that moment must be neutralized by the constraints.

The weight of a heavy body is distributed in all its parts, and will be shown to form a system of forces whose resultant is approximately a force at the centre of mass. In so far as this is true, the action of weight alone must produce pure translation.

Let XY be a vertical plane for the origin, with x directed vertically downward. For the centre of mass of a given body assume $\bar{z} = 0$, $\bar{v}_z = 0$, initially. Let the weight-acceleration g be regarded as parallel to x everywhere in the body, and constant in magnitude. This is very approximately true for small vertical range, and for bodies of ordinary size; the limits will of course depend upon the percentage accuracy

desired. The weight of any differential mass is $dP_x = g\,dm$; with $dP_y = 0$, $dP_z = 0$. The force of the resultant is then $\widehat{R} = m_1 g$ and parallel to x; hence $\dfrac{d^2\bar{z}}{dt^2} = 0$, and with the assumed initial conditions the path of the centre of mass lies in the XY plane. In calculating the moments, take axes X_1, Y_1, Z_1, at the centre of mass, and parallel to the reference-axes. The total moment $M(x_1) = 0$, because all the forces are assumed parallel to X. For the other axes we have, since y_1 and z_1 are the lever-arms to the order of approximation used,

$$M(y_1) = g\int_0^{m_1} z_1\,dm = 0; \quad M(z_1) = -g\int_0^{m_1} y_1\,dm = 0.$$

The announced result follows from these equations. On account of this property of the centre of mass, it is called the "centre of gravity."

86. The constraints required in order to restrict motion to each of several important types have been successively formulated in the preceding sections. Having been brought thus to the narrowed conditions of exclusive translation, it is a natural conclusion to the series in this direction to ask, concerning a state of rest, what is the criterion of its continuance under the action of given external forces. Granting that the vanishing of moment for every axis through the centre of mass is sufficient to prevent the setting up of motion relative to that point, an additional condition that the force of the resultant is zero indicates that the centre of mass has no acceleration, and can consequently never depart from rest. When the resultant velocity and acceleration are zero everywhere in the body, it is said to be in **equilibrium**. The conditions for equilibrium are therefore (the body being at rest)

$$\widehat{R} = 0; \quad \widehat{M} = 0. \tag{1}$$

Since the vanishing of a vector quantity is a necessary consequence, if three rectangular components of it are zero, the equations of equilibrium can be written,

$$P_x=0;\ P_y=0;\ P_z=0;\ M(x_1)=0;\ M(y_1)=0;\ M(z_1)=0. \quad (2)$$

The first three of equations (2) are summarized in the statement: In order that equilibrium may exist, the polygon of external forces must be closed. But composition of forces by means of the polygon construction is legitimate only when forces act at the same point. If extended in application to the present problem, it must be preceded by the supposed transfer of all forces to the centre of mass. Further the polygon construction, as a graphical process, becomes impracticable where the force-elements are differentials. In practical use, however, recourse is had to the formation of partial resultants, when external actions upon a body are localized upon comparatively small areas (cf. § 80, end); and, interpreted in this way, the polygon of forces becomes a valid and practicable scheme.

The group of conditions as regards moment is also written and used practically in a form that requires explanation. First, if $\widehat{R}=0$, the moments for all parallel axes become equal, because the possible forces form couples (and cf. § 80). Secondly, the idea of partial finite resultants acting in lines is utilized; of these there will be a finite number, and the summation sign (Σ) replaces the integral sign. Let any such partial resultant have components X, Y, Z, and act at the point (x, y, z). Then, as the moment equation of equilibrium, we must have for every possible set of rectangular reference-axes

$$\left.\begin{aligned} M(x) &= \Sigma\,(Zy - Yz) = 0, \\ M(y) &= \Sigma\,(Xz - Zx) = 0, \\ M(z) &= \Sigma\,(Yx - Xy) = 0. \end{aligned}\right\} \quad (3)$$

Equations (3) are generally found replacing the corresponding equations of group (2).

The forces originally applied may not of themselves fulfil the conditions $\widehat{R}=0$; $\widehat{M}=0$. Then, in order to establish equilibrium, constraints must be applied; either force with moment, or one element only of the two. The constraints under these circumstances are called the **equilibrant** of the other system. It is evident that any system of forces, and the equilibrant of that system, are equivalent in all but sign. Either system is the resultant of the other, with the sign changed for both force and moment, if both exist.

CHAPTER VI

HARMONIC MOTION AND PENDULUMS

87. When there is physical connection (visible or invisible) among a group of masses, the system often exhibits a tendency toward one particular configuration of its parts — the position of equilibrium. It then offers resistance to any attempted disturbance of this unconstrained or "natural" adjustment; and when displacement has been produced, there may be vibration or oscillation afterward about the equilibrium position. The word **displacement** is here used in a somewhat special sense, to denote for each point in the system a vector drawn to its disturbed position from that which it would occupy if equilibrium existed. Familiar examples of such oscillations are found in a clock pendulum, a tuning-fork, a compass needle, and a mass suspended from a spiral spring whose axis is vertical.

The phenomena in many branches of physics give evidence of forces made active by change of configuration that are at each part of the system opposite in direction to the displacement, and accurately or approximately proportional to its magnitude. Such a law connecting those quantities, or others that are closely analogous to them, is an assumed basis underlying the theoretical treatment of Elasticity, Sound, Light, and Electricity. The scope of this chapter, though, does not extend beyond application to some visible motions, where the law is fulfilled with reasonable approximation in bodies of finite size. For the most simple case we shall develop the essential features as consequences of the kinematical relations; and it will be shown subsequently that several common

forms of pendulum-motion can be made to depend upon equations of the same mathematical type.

Let a point O be taken as origin in the fixed line OX; and let the centre of mass C of a rigid body with mass m_1 have its acceleration and initial velocity in that line, the motion being one of translation. Then the path of C will be confined to the line OX; let the origin be its equilibrium position. If the resultant is a force proportional to the displacement of C, and always directed from that point toward O, we may write the equation of motion

$$P_x = -a\bar{x}. \tag{1}$$

The factor a is a positive constant that gives in magnitude the force per unit length of displacement. The equation shows force and displacement to be oppositely directed, for both positive and negative values of the latter.

Equation (1) is representative and important for a range of conditions wider than the preliminary statement. First, it enables us to trace the motion of the centre of mass in other cases than translation, so long as it expresses the force of the resultant in direction and magnitude. This equation can be regarded as complete for the discussion of such cases in proportion as the motion relative to the centre of mass is negligible. The couple-moment of the resultant is then omitted from consideration, the body being treated as a particle (§ 61, I). Secondly, the same equation applies where the centre of mass is constrained to follow a curved path, if P_x can be taken as the tangential force acting at C, while $\bar{x}$ is measured along the path from the equilibrium position. The same reservation as to the completeness of the statement is to be made here.

88. On dividing equation (1), § 87, by m_1, the resulting expression is essentially kinematical, and can be stated in the form

$$\frac{d^2x}{dt^2} = -kx. \tag{1}$$

Here x has been substituted for $\bar{x}$; and $\frac{a}{m_1}$ is replaced b k, the *acceleration* per unit length of displacement. The general consequences of the following discussion can be interpreted dynamically after restoring the proper mass-factors. Any point Q is said to execute **simple harmonic motion** about O, if it moves in OX according to equation (1), which shows the type of what are termed **harmonic equations.**

As Q moves away from O in either direction, its velocity and acceleration are of opposite sign; it must therefore come to rest. It will next move toward O, the speed becoming numerically greatest at the origin; it vibrates or oscillates about O as a centre, and the motion is evidently symmetrical with reference to that point. Let the velocity at O be $\pm v_1$; and the extreme distance from the centre be $\pm x_1$; the latter is called the **amplitude** of the vibration. Describe a circle with radius x_1 around the centre O, and let a point R (Fig. 19) move in it so that the radius OR turns with constant angular velocity ω. The acceleration of R is $-\omega^2 x_1$, and being proportional to RO it may be represented by that line. Consequently the projection of RO upon X represents the acceleration of R parallel to X, which is therefore $-\omega^2 x$, x being the coördinate of R. Provided that we take $\omega^2 = k$, it is always possible to make Q the projection of R, and regard simple harmonic motion in a line as a projection of uniform circular motion. If Q is to be continually the projection of R, the points must leave

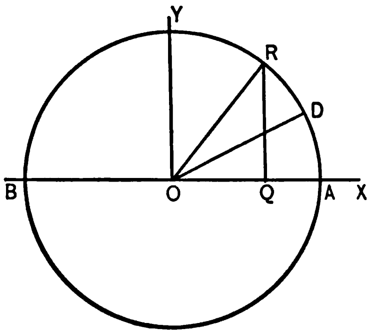


FIG. 19

A at the same instant. At that position the velocity parallel to X is zero for each point; and these velocities remain equal because p_x has always the same value for both points if $\omega^2 = k$. Then the centre of the circle will be the centre of the vibration, the radius will be the amplitude, and the square of the angular velocity will be the constant of the harmonic equation (cf. § 143).

Conveniently for many purposes time is assumed zero for the position A. Then the angle $ROA = \omega t$, increasing indefinitely with the time as Q goes on repeating its vibration. The position of Q at any time is determined by the equation

$$x = x_1 \cos(\omega t). \tag{2}$$

Its velocity is

$$v_x = -\omega x_1 \sin(\omega t) = -\omega y = -\sqrt{k}y. \tag{3}$$

On reaching the position O from A, the values are

$$x = 0;\quad v_1 = -\omega x_1 = -\sqrt{k}x_1. \tag{4}$$

Equations (1) and (2) are valid for all positions of R and Q, since v_x changes sign with y, and $\cos(\omega t)$ passes through zero at O.

89. The time required for Q to move from A back again to A is especially to be noted; this is equal to the time in which R moves over one entire circumference, reckoned from any starting-point. During this time Q completes one of its cycles, or sets of values for position, velocity, and acceleration. We find

$$T_1 = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{k}}. \tag{1}$$

T_1 is termed the **period** of the vibration; and the number of periods in one second, which is the reciprocal of T_1, is the **frequency**. Another prominent value of the time is that which

elapses while Q passes between A and B, or between any two positions corresponding to motion of R over a semicircumference; it is $\frac{T_1}{2}$. We have then

$$t_1 = \frac{T_1}{2} = \frac{\pi}{\omega} = \frac{\pi}{\sqrt{k}}. \tag{2}$$

The period of a simple harmonic motion is seen to be independent of the amplitude. It is clear that the time of passing once round a circle depends upon the given angular velocity, and not upon the radius. Consequently the constant k determines the period for Q, whatever amplitude may be given to the vibration. Such oscillations are **isochronous.**

The time in which Q moves from any position x' to any other x'', is the same as that in which OR moves through the corresponding angle. The relation can be expressed in the form,

$$x'' - x' = x_1[\cos(\omega t'') - \cos(\omega t')]. \tag{3}$$

The values of x'' and x' are equal if

$$t'' + t' = nT_1; \text{ or } t'' - t' = nT_1;$$

n being any whole number. Under the first supposition the two positions of Q coincide, but the speeds are equal with opposite signs; the two positions of OR are symmetrically related to OA. In the second case all the elements of the motion are exactly repeated. These results can be seen at once analytically, or from Figure 19.

The situation at any instant of Q in its cycle is spoken of as its **phase.** This is conveniently specified in terms of the difference of direction between OR and OA; ROA is then the phase-angle. The point Q returns to the same phase, whatever that may have been, after an interval T_1. If we assume $t = 0$ at A, the phase-angle is ωt (the time lying between zero

and T_1); if we take $t = 0$ at some other position of OR such as OD, the angle $DOA = \epsilon$ is called the epoch-angle, and the phase-angle is $\omega t + \epsilon$. Difference of phase between two positions is described in fractions of a period, or of the corresponding angle 2π. Thus the (smallest) difference in phase between A and O is $\frac{1}{4}$, meaning $\frac{T_1}{4}$, or $\frac{2\pi}{4} = \frac{\pi}{2}$. When the interval between two positions is t_1, they are frequently said to be in opposite phase; the first velocity (if not zero) has been reversed.

90. The discussion may next be extended by supposing that the initial velocity of the centre of mass C has any direction and magnitude, while the force of the resultant is proportional to the distance $\bar{r}$ of C from the origin, and has the direction CO. At $t = 0$, let $\bar{v} = \bar{v}_0$, and $\bar{r} = \bar{r}_0$; the motion of C will ensue in the plane determined by $\bar{v}_0$ and $\bar{r}_0$, since there is neither velocity nor acceleration perpendicular to that plane. Making use of polar components, we can apply the equations of motion for the centre of mass (§ 61, I), giving

$$P_r = m_1\left(\frac{d\bar{v}_r}{dt} - \bar{r}\omega^2\right) = -a\bar{r}; \tag{1}$$

$$P_\gamma = m_1\left(2\,\omega\bar{v}_r + \bar{r}\frac{d\omega}{dt}\right) = \frac{m_1}{\bar{r}}\frac{d}{dt}(\bar{r}^2\omega) = 0. \tag{2}$$

For the present conditions, too, the resultant is fully represented in these equations, if the motion be translation: this is compatible with a curved path for C (§ 18). And in motion of more general character it may be a sufficient approximation to treat the body as a particle (cf. § 87). We shall divide the equations of motion by m_1, so that they become kinematical in form as in § 88, and consider the characteristic motion and

path for any point Q whose accelerations satisfy them. The equations may be written,

$$\frac{dv_r}{dt} = r(\omega^2 - k); \tag{3}$$

$$\frac{d\omega}{dt} = -\frac{2\,\omega v_r}{r}. \tag{4}$$

We can infer from equation (2) that $r^2\omega$ is constant and equal to $r_0^2\omega_0$. Because r^2 is positive, ω cannot change sign; the radius-vector must continue to revolve in the same sense about O. Further, $r^2\omega$ is twice the rate at which area is described round the origin by r. Consequently that rate is constant here, or in any other case where $P_\gamma = 0$, and the force is **central**; *i.e.* continually directed toward a fixed centre, according to any law for magnitude.

Let the initial conditions be such that v_r is positive. If $\omega_0^2 < k$, $\frac{dv_r}{dt}$ is at first negative, and must remain so while $v_r > 0$. For then ω and $\frac{d\omega}{dt}$ have opposite signs by equation (4); and since ω cannot pass through the value zero, ω^2 must become continually smaller. If $\omega_0^2 > k$, $\frac{dv_r}{dt}$ is at first positive; but as ω^2 must decrease, the condition $\frac{dv_r}{dt} < 0$ is reached as before. Hence v_r must always be reduced to zero and become negative, giving for r a maximum value when $v_r = 0$. Again, let v_r be negative initially. If $\omega_0^2 > k$, $\frac{dv_r}{dt}$ is at first positive, and must remain so while $v_r < 0$. For ω and $\frac{d\omega}{dt}$ have then the same sign, and ω^2 must continually increase. If $\omega_0^2 < k$, we have $\frac{dv_r}{dt} < 0$ at first, but ω^2 increases until $\frac{dv_r}{dt}$ becomes positive as before. Consequently v_r must here be reduced to zero and then become positive, giving for r a minimum value when $v_r = 0$.

From equation (1) we have for any rectangular axes X and Y intersecting at O,

$$\frac{d^2x}{dt^2} = -kx; \quad \frac{d^2y}{dt^2} = -ky. \tag{5}$$

Hence the projections of Q execute simple harmonic motions about O with period $\frac{2\pi}{\sqrt{k}}$. Assume X particularly in the direction of a maximum or minimum for r, and let $t = 0$ when r and X coincide. At that instant $x = x_1$, $v_x = 0$, $y = 0$, $v_y = \pm\sqrt{k}y_1$. The integrals of equation (5) subject to these conditions are

$$v_x = \pm\sqrt{k(x_1^2 - x^2)}; \quad v_y = \pm\sqrt{k(y_1^2 - y^2)}. \tag{6}$$

In the region of positive x and y, v_y and v_x have opposite signs, giving

$$\frac{v_y}{v_x} = \frac{dy}{dx} = -\sqrt{\frac{y_1^2 - y^2}{x_1^2 - x^2}}. \tag{7}$$

This is the differential equation of an ellipse referred to its major and minor axes. The path of Q will be some ellipse with centre at O, whose eccentricity is fixed by the initial conditions; it will be traversed once in the period $\frac{2\pi}{\sqrt{k}}$. This is **elliptical harmonic motion,** and its components parallel to the axes of the ellipse are simple harmonic motions with equal period, differing by $\frac{1}{4}$ in phase.

If the initial velocity v_0 is perpendicular to the radius-vector, r_0 will lie in the major axis of the ellipse if $\omega_0^2 < k$; and in the minor axis if $\omega_0^2 > k$. With v_0 perpendicular to r_0, and $\omega_0^2 = k$, $\frac{dv_r}{dt} = 0$; then v_r and $\frac{d\omega}{dt}$ remain zero; the path is a circle described with uniform speed $v_0 = r_0\omega_0$. It is profitable to consider the corresponding dynamical conditions in connection with equation (1). For any given values of $\bar{r}_0$ and ω_0, the radial force necessary in order to make the path of C a circle is $-m_1\bar{r}_0\omega_0^2$. When the central force is numerically less than

this, it is insufficient to produce the constraint; the path diverges outward from the circle, and ω (supposed positive) grows less. But $m_1\bar{r}_0\omega_0^2 > a\bar{r}_0$ is equivalent to $\omega_0^2 > k$. Should the central force be greater than $m_1\bar{r}_0\omega_0^2$, it is in excess of the constraint necessary; the path will curve inside the circle, and ω increases. Then $m_1\bar{r}_0\omega_0^2 < a\bar{r}_0$, and $\omega_0^2 < k$. In either case there will be radial velocity when the circle is reached that corresponds to exact adjustment between constraint necessary and force supplied. The elliptical path can be regarded as due to oscillation about a circle of stable motion, whose diameter is intermediate between the major and minor axes.

91. Both simple and elliptical harmonic motions result from the action of a central force proportional to displacement, unhindered by any resistance. In the motion next to be investigated it is supposed that the "driving force" still acts according to the harmonic law, but its effect is modified by a resistance that is proportional to the actual speed. This type is important from a physical point of view; it is of frequent occurrence, because friction between a body and air or other substance can often be assumed to vary as their relative speed, so long as this does not exceed a limiting value. The oscillations of a magnet, when influenced by induced currents due to its motion, can also be brought into this scheme. The fundamental equation will be written as before for the centre of mass of a rigid body; and it is subject to the same considerations as regards the completeness of its statement for translation and other types of motion.

Let the origin be O, and consider motion of C in any fixed line OX. The conditions indicated above are expressed in the equation of motion,

$$m_1 \frac{d^2\bar{x}}{dt^2} = -\left(a\bar{x} + a_1 \frac{d\bar{x}}{dt}\right). \tag{1}$$

Here a is again central force per unit of displacement; and a_1 is force per unit speed developed in resistance to the actual motion. Examination in detail shows that there are the proper changes of sign in $\bar{x}$ and $\frac{d\bar{x}}{dt}$ to accommodate the equation to all phases of a possible vibration about O. Divide equation (1) by m_1; place $\frac{a}{m_1} = k$, and $\frac{a_1}{m_1} = 2\,k_1$; and let the coördinate be general, to express kinematical relations for any point Q (Fig. 20) whose motion satisfies the equation. We obtain

$$\frac{d^2x}{dt^2} = -\left(kx + 2\,k_1\frac{dx}{dt}\right), \tag{2}$$

k and k_1 being positive constants. It has been shown in § 88 that simple harmonic motion is a projection of uniform circular motion. With a certain limitation to be discussed later (§ 92), the point Q, whose motion is represented by equation (2), can be considered as a projection of a point R that moves according to a comparatively simple law, which we now proceed to deduce.

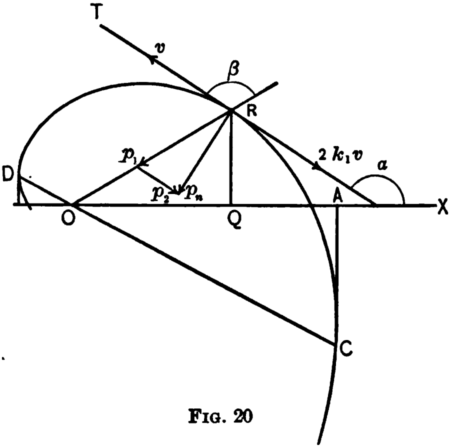


FIG. 20

Assume O as pole, and OX as reference-line; consider Q in the position shown, and moving toward O. It is clear that x must be the projected radius-vector, and $\frac{dx}{dt}$ the projected velocity of R. Hence that point has component accelerations $-kr$ (inward along r), and $-2\,k_1v$ (backward in the

tangent), the sum of whose projections constitutes the second member of equation (2); while v has some such direction as RT. Let v make an angle β with r, and $\alpha = \gamma + \beta$ with X. For any path of R in the plane of the diagram,

$$p_t = \frac{dv}{dt}; \quad p_n = -v\frac{d\alpha}{dt}. \tag{3}$$

Decomposing p_n into oblique components in the lines of r and v, we obtain the component accelerations p_1 and p_2 in those lines for R, on adding together the two elements that fall in the tangent. Since $v = r\omega \operatorname{cosec} \beta$, the components are in magnitude

$$p_1 = \omega \operatorname{cosec}^2 \beta \frac{d\alpha}{dt} r; \tag{4}$$

$$p_2 = v\frac{d\alpha}{dt} \cot \beta + \frac{dv}{dt}. \tag{5}$$

In order to make p_1 equal to kr, we must have

$$\omega \operatorname{cosec}^2 \beta \frac{d\alpha}{dt} = k. \tag{6}$$

This condition can be fulfilled if both ω and β are constant. Then $\frac{d\alpha}{dt} = \omega$; $\frac{dv}{dt} = \frac{dr}{dt}\omega \operatorname{cosec} \beta = \omega v \cot \beta$; and

$$p_2 = 2\,\omega v \cot \beta. \tag{7}$$

So that the assumed condition makes p_2 equal to $2\,k_1 v$, if we take $k_1 = \omega \cot \beta$ (numerically). The path of R is consequently an equiangular or logarithmic spiral; and R moves in the curve with such a speed that OR turns with constant angular velocity ω. The diagram is drawn to scale for $\cot \beta = -0.5$. If Q is initially the projection of R, it will remain so, provided that v_x for both points has the same value

at $t=0$; the constants having been so adjusted that the accelerations parallel to X are equal. At $t=0$, let $v_x=0$; then

$$\gamma_0+\beta=\frac{\pi}{2},\ r=r_0,\ x_0=r_0\cos\gamma_0.$$

For any value of t,

$$\gamma=\gamma_0+\omega t;\ r=r_0 e^{\omega t\cot\beta}. \tag{8}$$

With positive ω it is evident that $\beta>\frac{\pi}{2}$, and hence $\cot\beta<0$, in this application of the spiral; so we may write $r=r_0e^{-k_1t}$.

92. It is not possible to connect such a spiral motion directly with an unlimited range of values for k and k_1, because the ratio

$$\frac{k_1^2}{k}=\frac{\omega^2\cot^2\beta}{\omega^2\operatorname{cosec}^2\beta}=\cos^2\beta; \tag{1}$$

and accordingly the restriction $k_1^2\leqq k$ is imposed. The motion of Q is to be traced first under the supposition $k_1^2<k$, reserving the values $k_1^2 \gtreqless k$ for special discussion. The coördinate of Q is

$$x=r\cos(\gamma_0+\omega t); \tag{2}$$

and its velocity,

$$v_x=v_r\cos(\gamma_0+\omega t)-v_\gamma\sin(\gamma_0+\omega t). \tag{3}$$

When $$v_x=0,\ \operatorname{tg}(\gamma_0+\omega t)=\cot\beta. \tag{4}$$

This equation is satisfied for $t=0$, $\frac{\pi}{\omega}$, $\frac{2\pi}{\omega}$, etc., when R crosses the line DC. The corresponding values for x are, writing t_1 for $\frac{\pi}{\omega}$,

$$\left.\begin{array}{c} x_0=r_0\cos\gamma_0;\ x_1=-r_1\cos\gamma_0=-r_0e^{-k_1t_1}\cos\gamma_0;\\ x_2=r_0e^{-2k_1t_1}\cos\gamma_0;\ \text{etc.}\end{array}\right\} \tag{5}$$

The sign of x is alternately plus and minus; Q vibrates about O. The period at the end of which r returns to its original

direction, or Q comes to rest again on the same side of the origin, is

$$T_1 = 2\,t_1 = \frac{2\,\pi}{\omega} = \frac{2\,\pi}{\sqrt{k - k_1^2}}. \tag{6}$$

Were the resistance to be removed, leaving free play to the central force, the period would be

$$T_0 = \frac{2\,\pi \sin\beta}{\omega}. \tag{7}$$

One effect of the resistance, therefore, is to lengthen the period in comparison with that due to the central force alone, the ratio being $\frac{T_0}{T_1} = \sin\beta$. But the vibration remains isochronous. The amplitude or elongation diminishes continually in magnitude, the ratio of distances from O at successive turning-points ($v_x = 0$) being constant. Thus,

$$\frac{x_1}{x_0} = \frac{x_2}{x_1} = e^{-k_1 t_1} = e^{\pi \cot\beta}. \tag{8}$$

The factor $e^{-k_1 t_1}$ is known as the damping factor, and the exponent $k_1 t_1$ as the (natural) logarithmic decrement, of the oscillation.

There is no such narrow limit physically, however, as that hitherto assigned to the relative magnitudes of k and k_1; for example, a body may be immersed in a strongly viscous liquid like glycerine or tar, while acted on by a weak spring. Under such circumstances a_1 may be larger than a, giving $k_1^2 > k$ (eq. (1) and (2), § 91). It becomes necessary to examine how the motion of our representative point Q is then modified. We shall begin by looking into the boundary case $k_1^2 = k$. Let the rectilinear motion occur in a given medium, so that k_1 is fixed; the central force being at first such that $k > k_1^2$. Let us trace the consequences of weakening the central force, so that k decreases toward equality with k_1^2. The angle β approaches π (eq. (1)), $\cot^2\beta$ increases, and $\omega^2 = \left(\frac{k_1}{\cot\beta}\right)^2$ decreases, both

without limit. The period $T_1 = \dfrac{T_0}{\sin\beta}$ also increases without limit; $\sin\beta$ approaches zero, while T_0 lengthens as k becomes less. The general effect, therefore, is to give indefinitely great values for t_1, as $\dfrac{k_1^2}{k}$ approaches unity. The boundary value of equation (2) is

$$x = r_0 e^{-k_1 t}\cos\left(\omega t - \frac{\pi}{2}\right) \quad \left[\beta = \pi,\ \gamma_0 = -\frac{\pi}{2}\right]. \tag{9}$$

Consequently the limiting value for the amplitude at $\omega t_1 = \pi$ is

$$x_1 = r_0 e^{-k_1 t_1}\cos\left(\frac{\pi}{2}\right) = 0 \qquad [t_1 = \infty]. \tag{10}$$

According to this equation, the boundary condition of affairs is that x_1 ceases to be negative, and Q approaches the origin asymptotically. The oscillatory character of the motion becomes evanescent; it is now aperiodic or "dead beat." On account of the exponential factor, x is imperceptibly different from zero for moderate values of the time. The same conclusions can be otherwise reached by connecting the discussion of the limiting case with the condition $\gamma_0 = 0$, and beginning the time interval when the curve crosses the X axis at $r = r_0$.

Further, the type of motion just described is not essentially altered when $k_1^2 > k$, for the resistance is *passive;* it appears only in proportion as motion toward the origin has been actually produced. The active tendency toward the equilibrium position continues to manifest itself until the central force vanishes at the origin ($t = \infty$). Any values of the constants can be expressed under the indeterminate forms,

$$k_1 = 0 \cdot \infty; \quad k = \frac{0}{0}.$$

This may be regarded as an indication that the relation between k and k_1 is dissolved at the limit.

93. Let a cycloid be fixed with its plane vertical, the vertex being its lowest point; and let the centre of mass C of a rigid body whose mass is m_1 be constrained to move along the curve. We shall first count all frictional resistances as negligible, and the tangential force, supposed to act at C, is regarded as due entirely to the weight $m_1 g$ of the body. In this instance and all similar ones, unless the contrary be noted, weight in vacuum is to be understood; g is the true (local) value of the weight constant, or acceleration due to weight. The equilibrium position of C is at the vertex, which shall be taken as origin, with Y vertically upward. When C is at any point (x, y) of the cycloid, its equation of motion in the tangent is

$$-m_1 g \frac{dy}{ds} = m_1 \frac{d^2 s}{dt^2}. \tag{1}$$

For the cycloid with axes chosen as above, r_1 being the radius of the generating circle, $s^2 = 8\, r_1 y$. Hence $\dfrac{dy}{ds} = \dfrac{s}{4\, r_1}$, and substitution in (1) gives, omitting the factors m_1,

$$\frac{d^2 s}{dt^2} = -\frac{g}{4\, r_1} s. \tag{2}$$

This agrees in type with the harmonic equation (1), § 88. The point C will therefore execute isochronous vibrations about the vertex, its motion along the cycloid being simple harmonic. The period is

$$T_1 = 2\,\pi \sqrt{\frac{4\, r_1}{g}} = 4\,\pi \sqrt{\frac{r_1}{g}}. \tag{3}$$

If we now include in the scheme a resistance opposite to the velocity and proportional to its magnitude, equation (2) becomes

$$\frac{d^2 s}{dt^2} = -\frac{g}{4\, r_1} s - 2\, k_1 \frac{ds}{dt}. \tag{4}$$

This is of the same mathematical type as equation (2), § 91, with s replacing x, and $\frac{g}{4r_1}$ as a particular value of k. If then $k_1^2 < \frac{g}{4r_1}$, the motion will be a damped vibration of period

$$T_1 = \frac{2\pi}{\sqrt{\frac{g}{4r_1} - k_1^2}}. \tag{5}$$

When $k_1^2 \overline{\gtrless} \frac{g}{4r_1}$ the motion is aperiodic; C approaches the vertex asymptotically from any disturbed position. The other details of the previous solution can be paralleled throughout.

94. When forces or force-moments are used to produce torsional strain within a system, and then removed, we may find vibrations resulting under the influence of a moment due to the strain itself. These effects are noticeable with various arrangements, and with materials ranging from steel wire to silk fibre; but we shall be able to gain sufficient insight into the general conditions of such action by considering one simple example as a model. Let a rigid body AB (Fig. 21), suspended by a vertical wire DE that is firmly clamped at the upper end D, be twisted about the axis of the wire from the equilibrium position and then released. The body is supposed to have a horizontal plane of symmetry; and to be rigidly attached to the wire, with the centre of mass C in the prolongation of the axis (cf. § 83, III). The parts of the body will then oscillate in horizontal planes about their positions of equilibrium if the tor-

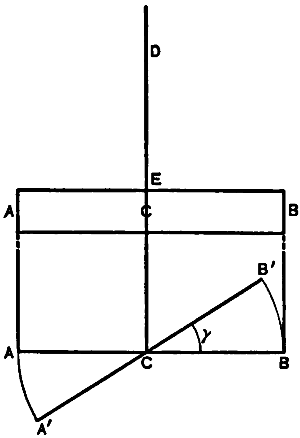


FIG. 21

sional moment alone acts. That quantity depends, in ways not specially to be discussed here, upon the length, material, etc., of the wire. When those elements are given, the experimental results prove the angular displacement of the body and the torsional moment acting upon it to be proportional magnitudes of opposite sign. This law seems to be fulfilled accurately through a wide range of displacement. The lower part of the diagram shows the vibrating body in a horizontal section through C. AB is the equilibrium position of any line; $A'B'$ its displaced position at any time; $B'CB = \gamma$. Provided that the moment of inertia is effectively that of the attached body (I_C), the experimental facts justify the equation

$$-k\gamma = I_C \frac{d^2\gamma}{dt^2} \tag{1}$$

(cf. eq. (1), § 73); k being a positive constant expressing the force-moment per unit of angular displacement. This equation is plainly harmonic for the variable γ, with period

$$T_1 = 2\pi\sqrt{\frac{I_C}{k}}. \tag{2}$$

The constant k of such a torsion pendulum is not sensibly affected by moderate changes in the weight of the body attached; and this fact is the basis of an experimental method by means of which the moment of inertia I_C and the torsion constant of the suspension can be determined. Let a known mass be added to the body AB, and so adjusted in position that the centre of mass still lies in the axis DE. Care is of course exercised in making the auxiliary mass accurately symmetrical, and of homogeneous material, so that its moment of inertia I'_C can be calculated from its dimensions. If the limit for unchanged k has not been exceeded, the new period is

$$T'_1 = 2\pi\sqrt{\frac{I_C + I'_C}{k}}. \tag{3}$$

T_1 and T'_1 being found by observation, equations (2) and (3) yield values for I_C and k.

95. The ordinary form of clock pendulum is essentially a rigid body rotating about a fixed horizontal axis under the influence of its weight. In order to determine the most important characteristics of the motion of such a weight pendulum, we shall eliminate from consideration all secondary elements of the problem. These may be introduced afterward to any extent desired, as "corrections" applied to the main result. Let the diagram (Fig. 22) represent a vertical section of the body, perpendicular to the horizontal rotation-axis shown at O, and containing the centre of mass C. The point O is commonly known as the centre of suspension; it shall be the origin, X being drawn vertically downward. The mass of the body is m_1; the distance OC is $\bar{r}$; and the angle γ (COX) is measured from the equilibrium position of C. The total moment of the weight is $-m_1 g\bar{r}\sin\gamma$ (§ 85), and the general equation ((1), § 73) adapted to this special case becomes

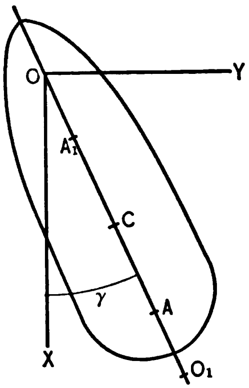


FIG. 22

$$-m_1 g\bar{r}\sin\gamma = I_O\frac{d\omega}{dt}. \tag{1}$$

The angular velocity at any position can be obtained by direct integration of (1), writing $\frac{\omega\,d\omega}{d\gamma}$ for $\frac{d\omega}{dt}$; or the work equation ((2), § 68) can be applied. By either method the result is

$$m_1 g\bar{r}(\cos\gamma - \cos\gamma_0) = \frac{\omega^2 - \omega_0^2}{2}I_O, \tag{2}$$

the initial conditions being $\gamma=\gamma_0$, $\omega=\omega_0$. In discussing pendulum motion it is usually convenient to choose $\omega_0=0$ at $t=0$, in which case γ_0 is the (angular) amplitude.

A prominent element connected with pendulums is the time between successive positions of rest, the time of one beat. In the present instance the exact integration of equation (2), that would determine γ as a function of t, is impossible. A process of approximation being necessary, it may be begun at equation (1), which would be of harmonic type for the variable γ, if that angle itself appeared in the first member instead of its sine. But the difference $(\gamma-\sin\gamma)$ is a vanishing quantity as the angle diminishes. So we shall confine the solution to small values of γ, and set the limit of accuracy in the first approximation at the point of neglecting the difference between the angle and its sine in equation (1). It may then be written

$$\frac{d^2\gamma}{dt^2}=-\frac{m_1g\bar{r}}{I_o}\gamma \qquad [\sin\gamma=\gamma]. \tag{3}$$

The motion of C appears now as a simple harmonic vibration along its circular arc, for $s=r\gamma$, and r is constant. The time of beat and period are

$$t_1=\pi\sqrt{\frac{I_o}{m_1g\bar{r}}};\qquad T_1=2\pi\sqrt{\frac{I_o}{m_1g\bar{r}}}. \tag{4}$$

To this approximation the oscillations are isochronous. However, since $\gamma>\sin\gamma$ in magnitude, and the difference varies with the angle, the true period T_0 depends upon amplitude, and is greater than T_1.

96. We return to equation (2), § 95, as a starting-point for closer approximation to the true period. The initial values being $t=0$, $\omega_0=0$, $\gamma=\gamma_0$, we obtain, in terms of the half-angle,

$$\frac{1}{2}\sqrt{\frac{I_o}{m_1g\bar{r}}}\frac{d\gamma}{dt}=\pm\sqrt{\sin^2\frac{\gamma_0}{2}-\sin^2\frac{\gamma}{2}}. \tag{1}$$

The true time of beat is twice the time in which C moves from its extreme position to the vertical OX, so we may write for the first half-period, during which the angular velocity is negative,

$$t_0 = \sqrt{\frac{I_O}{m_1 g \bar{r}}} \int_0^{\gamma_0} \frac{d\gamma}{\sqrt{\sin^2 \frac{\gamma_0}{2} - \sin^2 \frac{\gamma}{2}}}. \tag{2}$$

Confining our attention to the integral, we first substitute a more convenient variable ϕ, connected with γ by the equation

$$\sin \frac{\gamma}{2} = \sin \frac{\gamma_0}{2} \sin \phi. \tag{3}$$

The form of function is so chosen as to recognize the fact that γ is never greater than γ_0 during the interval. On changing the limits of the integration accordingly, and placing $\sin^2 \frac{\gamma_0}{2} = a^2$, for brevity, we have the result ready for expansion into a series:

$$\int_0^{\gamma_0} \frac{d\gamma}{\sqrt{\sin^2 \frac{\gamma_0}{2} - \sin^2 \frac{\gamma}{2}}} = 2 \int_0^{\frac{\pi}{2}} (1 - a^2 \sin^2 \phi)^{-\frac{1}{2}} d\phi. \tag{4}$$

Expressing the required integral as a sum by means of the series,

$$2 \int_0^{\frac{\pi}{2}} (1 - a^2 \sin^2 \phi)^{-\frac{1}{2}} d\phi$$

$$= 2 \left[\int_0^{\frac{\pi}{2}} d\phi + \frac{a^2}{2} \int_0^{\frac{\pi}{2}} \sin^2 \phi \, d\phi + \frac{1 \cdot 3}{2 \cdot 4} a^4 \int_0^{\frac{\pi}{2}} \sin^4 \phi \, d\phi + \text{etc.} \right]. \tag{5}$$

The integral $\int_0^{\frac{\pi}{2}} \sin^{2n} \phi \, d\phi$ is a well-known form whose value is $\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \frac{\pi}{2}$. The corrected time of beat is then

$$t_0 = \pi \sqrt{\frac{I_O}{m_1 g \bar{r}}} \left[1 + \frac{a^2}{4} + \frac{9}{64} a^4 + \text{etc.} \right]. \tag{6}$$

If all terms beyond the first are rejected, the previous approximation reappears. It is often sufficient to retain the first two terms. And if *in the correction* we place $\sin\frac{\gamma_0}{2}=\frac{\gamma_0}{2}$, we obtain

$$t_0=\pi\sqrt{\frac{I_O}{m_1 g\bar{r}}}\left[1+\frac{\gamma_0^2}{16}\right]. \tag{7}$$

At any stage of the approximation, of course, $T_0=2\,t_0$. The fraction $\frac{T_0-T_1}{T_1}=\frac{t_0-t_1}{t_1}$ is frequently tabulated as the "correction for amplitude." Its value for an elongation from the vertical $\gamma_0=\frac{\pi}{36}$ is about 0.0005.

97. In proportion as all dimensions of the oscillating body in planes perpendicular to the rotation-axis are small compared with OC, we approach the ideal of a **simple pendulum** of length $\bar{r}$ — a particle at C with massless connection to the axis. In contrast with this the real pendulum is termed **physical** or **compound.** Since $I_C=0$ is the limiting condition in order to consider the mass as a particle, the general relation for parallel axes $I_O=I_C+m_1\bar{r}^2$ (§ 150) becomes $I_O=m_1\bar{r}^2$, giving for the simple pendulum

$$\frac{d^2\gamma}{dt^2}=-\frac{g}{\bar{r}}\gamma;\qquad \text{and } t_1=\pi\sqrt{\frac{\bar{r}}{g}}\quad [\sin\gamma=\gamma]. \tag{1}$$

The mass factor does not appear in these equations; they may therefore be applied to a differential mass at any part of the body distant r from the axis. If such element of mass could be freed from the constraint caused by being embedded in the body, it would beat as a simple pendulum of length r in the time $t'_1=\pi\sqrt{\frac{r}{g}}$. Because $\bar{r}$ is an average value of r, it follows that by the constraint of moving as a rigid body the beat of some of these elementary pendulums is quickened, while that of others is made slower. Wherever $t'_1=t_1$ (eq. (4), § 95), no such

change would be caused by connecting that differential mass with the body. Denoting the corresponding particular value of r by l, and placing

$$\pi\sqrt{\frac{l}{g}} = \pi\sqrt{\frac{I_O}{m_1 g \bar{r}}}, \tag{1(a)}$$

we find

$$l = \frac{I_O}{m_1 \bar{r}} \tag{2}$$

as the equation of condition for $t'_1 = t_1$. The length l which satisfies equation (2) is that of a simple pendulum called *equivalent* to the physical pendulum. When that length is laid off in the direction OC, the point thus located is named the **centre of oscillation**; it is A in the diagram (Fig. 22). If the entire mass could be situated at this centre, the beat of the pendulum would not be changed. The series that measures the ratio $\frac{t_0}{t_1}$ is the same for both simple and compound pendulums (eq. (6), § 96).

We proceed to prove that any centre of suspension and its centre of oscillation are in conjugate relation; in other words, if the pendulum (Fig. 22) be reversed so that the rotation-axis is perpendicular to the diagram at A, making that point the centre of suspension, then O will be the corresponding centre of oscillation. This shall be done by showing that the length of the equivalent simple pendulum remains unaltered by the change of axis from O to A. Using the idea of radius of gyration (§ 75), equation (2) may be written,

$$l = \frac{I_O}{m_1 \bar{r}} = \frac{k_O^2}{\bar{r}} = \frac{k_C^2 + \bar{r}^2}{\bar{r}} = \frac{k_C^2}{\bar{r}} + \bar{r}. \tag{3}$$

Similarly for the axis at A,

$$l_1 = \frac{I_A}{m_1 \bar{r}_1} = \frac{k_A^2}{\bar{r}_1} = \frac{k_C^2 + \bar{r}_1^2}{\bar{r}_1} = \frac{k_C^2}{\bar{r}_1} + \bar{r}_1. \tag{4}$$

Hence $(l - \bar{r})\bar{r} = (l_1 - \bar{r}_1)\bar{r}_1$; since further $l = \bar{r} + \bar{r}_1$, $l = l_1$, and the proposition is proved.

From the form of equation (3) we can conclude that generators of a cylindrical surface whose axis passes through C are rotation-axes for which t_1 has equal values. Consequently there are four points in OC produced for which the equivalent simple pendulum has the same length. These are arranged symmetrically about C in pairs O, O_1; A, A_1; and O does not in general coincide with A_1, for the necessary relation $\frac{I_O}{m_1\bar{r}} = 2\bar{r}$ is not fulfilled. The length of l is either OA or O_1A_1. The "reversible pendulum" method of determining g depends upon locating such points as O and A, for which the times of beat about parallel axes are equal, while the distance between the points is the length of the equivalent simple pendulum.

98. In all uniplanar motion the total work of the forces acting may be expressed as the sum of two terms (§ 63). For instance, equation (2), § 95, can be put into the form,

$$m_1 g(\bar{x} - \bar{x}_0) = I_C \frac{\omega^2 - \omega_0^2}{2} + m_1 \frac{\bar{v}^2 - \bar{v}_0^2}{2}, \tag{1}$$

showing that in the motion there assumed the work of the weight is guided into two channels, and a definite division of energy is insured between translation with C and rotation about a certain horizontal axis through that point. The special feature of pure rotation is that $\bar{v} = \bar{r}\omega$ permanently (§ 80). The limiting condition of the simple pendulum ($I_C = 0$) makes the first term in the second member of equation (1) vanish, independently of ω. But that term will remain zero also under the condition $\omega = \omega_0$ always; that is to say, if the angular velocity about the axis at C is constant. Consequently we can obtain the same elimination of factors as we found for the simple pendulum, without the device of an imagined particle. The statement $t_1 = \pi\sqrt{\frac{\bar{r}}{g}}$ applies to any

mechanical contrivance such that the body has constant angular velocity of any value about the axis at C, while weight acts and that point is guided along the circle of radius $\bar{r}$ without appreciable resistance. The essential condition is that the couple-moment of the resultant shall be zero.

But equation (1) does not contain even implicit reference to a circle, for which we may substitute any *smooth* (plane) guide-curve if the condition $\bar{v} = \bar{r}\omega$ be abandoned. For example, the cycloidal motion of § 93 will proceed as there shown when accompanied by any constant rotation about an axis at C perpendicular to the diagram. The (curvilinear) translation previously considered appears now as a special case, with $\omega = \omega_0 = 0$. None of the work is diverted into energy of rotation so long as ω is constant at any value.

99. Weight serves to maintain oscillations in another form of pendulum, whose equation of motion may be reduced with reasonable approximation to the same type as equation (1), § 95. This is the **bifilar pendulum**, which we shall analyze under its simplest conditions. Let AD and BE (Fig. 23) be wires of negligible mass and stiffness. Their upper ends are fixed in the horizontal line DE; they are of equal length h, and parallel at the distance $2\,l$. The plane of the diagram is vertical, and the attached body ACB is supposed a regular solid (cylinder or rectangular prism) of mass m_1, with a plane of symmetry AB horizontal. The centre of mass C lies midway in the line AB. The upper part of the dia-

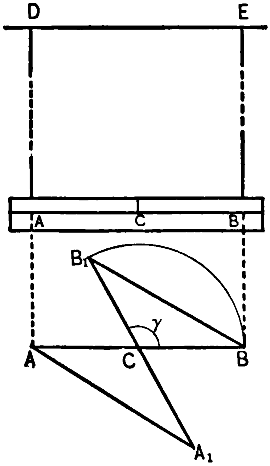


FIG. 23

gram represents the position of equilibrium, the wires being then vertical and the lines AB and DE parallel. If the body is turned about a vertical axis through C, so that AB takes the position A_1B_1 at an angle γ with AB, and is then released, there will be oscillations under the influence of a force-moment due to the weight of the body. We shall show that this moment is approximately proportional to the angle γ taken with reversed sign, when the displacement is small.

In any displaced position, the horizontal projections of the suspending wires are the chords AA_1, BB_1 (Fig. 23). Figure 24 represents the relations in the plane BEB_1; let $BEF = \beta$. On account of the symmetry in the arrangement the tensions T in the wires are equal, and their projections H into the horizontal plane through C constitute a couple. Rotation about the vertical axis is determined by this couple-moment; the other part of the resultant is a vertical force at C. The vertical projection of either tension being V, the vertical force is $(2V - m_1g)$, with the positive direction upward. For the force H and moment M of the couple we have, if a is the altitude of the isosceles triangle BCB_1,

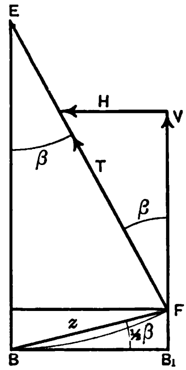


FIG. 24

$$H = T \sin \beta = V \operatorname{tg} \beta; \quad M = -2aV \operatorname{tg} \beta. \tag{1}$$

Equating expressions for the area of the same triangle, we find

$$\tfrac{1}{2} l^2 \sin \gamma = \tfrac{1}{2} a \cdot \overline{BB_1}; \quad 2a = \frac{2l^2}{BB_1} \sin \gamma. \tag{2}$$

In the usual construction of the pendulum, $\frac{h}{l}$ is a large ratio, with the consequence that β is a small angle, even for com-

paratively large values of γ; and $\sin\beta$ can replace $\operatorname{tg}\beta$ with inconsiderable error. Thus $\frac{h}{l} = 25$ gives for $\gamma = \pi$, $\beta < \frac{\pi}{36}$, and $\operatorname{tg}\beta - \sin\beta = 0.003$. But $\sin\beta = \frac{BB_1}{h}$; and introducing this factor,

$$M = -V\frac{BB_1}{h}\cdot\frac{2\,l^2}{BB_1}\cdot\sin\gamma = -V\frac{2\,l^2}{h}\sin\gamma = I_C\frac{d^2\gamma}{dt^2}. \qquad (3)$$

V is not strictly constant, though $V = \frac{m_1 g}{2}$ so long as the pendulum is held at a given displacement by horizontal forces. When motion ensues upon release there is vertical acceleration, so that V differs from $\frac{m_1 g}{2}$. In proportion as the range of γ is small and the ratio $\frac{h}{l}$ large, however, we can make a closer first approximation by considering V constant, as well as $\sin\gamma = \gamma$. It is worth notice that the substitution of γ for $\sin\gamma$ offsets in part the effect of writing $\sin\beta$ for $\operatorname{tg}\beta$. The approximate harmonic equation is

$$-m_1 g\frac{l^2}{h}\gamma = I_C\frac{d^2\gamma}{dt^2} \qquad \begin{bmatrix} \operatorname{tg}\beta = \sin\beta \\ V = \tfrac{1}{2}m_1 g \\ \sin\gamma = \gamma \end{bmatrix}. \qquad (4)$$

The corresponding period is seen to be

$$T_1 = 2\,\pi\sqrt{\frac{hI_C}{m_1 g l^2}}. \qquad (5)$$

This may be corrected for amplitude, *so far as γ is concerned,* by use of the series in equation (6), § 96.

100. Another common example of pendulum-motion is afforded by a magnet; its equilibrium position and period being determined at any time by its own condition and its surroundings conjointly. One characteristic quality of magnets is polarity; and the degree to which magnetic properties have

been developed in them is quantitatively expressed by their magnetic polarization (magnetic moment); we shall denote this by Q. The earth seems to exercise an influence upon any magnet that brings a particular direction in it (the magnetic axis) to coincide with the line of dip, when other forces are removed or neutralized. Magnetic moment is then naturally associated with the prominent line of polar quality — the magnetic axis, whose direction is assigned to Q, making that quantity a vector. Similarly the element external to the magnet, and representing its surroundings in the dynamical effect, has intensity that is suggestively connected with the line of dip. The direction of that line is accordingly adopted for the quantity called field, and here written F. Experiment has established as a fact that the action of the earth upon a magnet is sensibly a couple-moment whose axis and magnitude can be simply expressed by means of the vectors Q and F. If γ is an angle measured from F to Q, the couple-moment (about its own axis) is in magnitude and sign $-QF\sin\gamma$, becoming $-QF$ when $\gamma=\frac{\pi}{2}$, and zero when Q and F coincide in the position of equilibrium. Let the direction of F be assumed for X, and the plane determined by Q and F as XZ (Fig. 25). Then the couple-axis has the direction of Y while $\gamma<0$, and of negative Y while $\gamma>0$. When this couple is the complete resultant, the magnet will rotate about Y as a free axis (§ 83, III), provided that line is a principal axis for

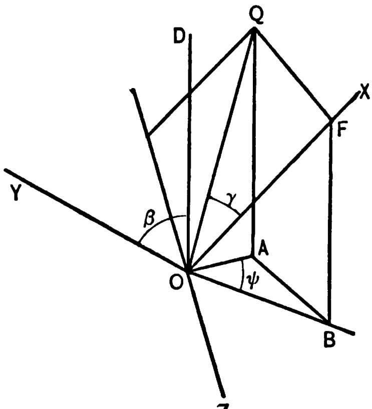


FIG. 25

its centre of mass. Where the magnet is constrained to rotate about some other axis, whose direction makes an angle β with the couple-axis, the effective couple-moment is $-QF \sin\gamma \cos\beta$ (§ 72). In dealing with couple-moments, which are independent of the positions of axes (§ 71), we are concerned with the directions merely of the latter.

We see from the diagram (Fig. 25), where OF and OQ represent the vectors F and Q, that $QF \sin\gamma$ is twice the area of OFQ. But we know that $OAB = OFQ \cos\beta$, if the former triangle is the projection of the latter upon a plane perpendicular to any rotation-axis OD. Hence, calling the projections of the vectors F and Q upon that plane F_1 and Q_1, and denoting by ψ the angle that Q_1 makes with F_1, we obtain the relation,

$$Q_1F_1 \sin\psi = QF \sin\gamma \cos\beta. \tag{1}$$

For any plane, then, the couple-moment about the normal is given in terms of the projected quantities and the angle between them, by an expression of the original type. Neither Q nor F is accurately constant; outside of narrow limits in space and time, the former may exhibit noticeable changes — at least in magnitude; and the latter is subject to changes and fluctuations in both direction and magnitude, some of which are neither slow nor minute. But for sets of observations completed within a short time at the same place, both vectors may in most cases be regarded as constant. The equation of motion,

$$-Q_1F_1 \sin\psi = I_0 \frac{d^2\psi}{dt^2}, \tag{2}$$

can then be integrated by the process of either § 95 or § 96, giving for time of beat,

$$t_1 = \pi\sqrt{\frac{I_0}{Q_1F_1}}; \quad t_0 = \pi\sqrt{\frac{I_0}{Q_1F_1}}\left[1 + \frac{a^2}{4} + \text{etc.}\right]. \tag{3}$$

Ordinarily the magnetic and rotation axes are perpendicular, which makes $Q_1 = Q$; in addition the rotation-axis is vertical in many physical instruments (galvanometers, magnetometers). Writing H for the particular value of F_1 under the latter condition, the characteristic factor of equation (3) takes the form,

$$\sqrt{\frac{I_0}{QH}}.$$

In the above discussion the earth alone has been considered as a source of magnetic field. Of course, the result can be adapted to other combinations in proportion as the field there is constant in direction and magnitude throughout the space traversed, and during the magnet's oscillations. The results of §§ 91, 92 find peculiarly useful application as a modification of the present scheme.

101. The results obtained in this chapter are of value enough in themselves to warrant deliberate discussion of the thoughts that are fundamental to them. They gain in significance, however, through their connection with other important matters. First, every student of physics should be aware of the interesting historical elements presented by the early investigations into the problems of the cycloidal and circular pendulums. Huyghens and Newton were pioneers in this field; but through their labors in it they were able to register such an advance in knowledge that they left every principle of first consequence established. They made original discovery of the laws of isochronous vibrations, the existence of the centre of oscillation, and the conjugate relation of the latter to the centre of suspension. And they accomplished these things, although handicapped by the cumbersome and difficult methods of proof which were then the only ones available. In connection with his part in the task, moreover, Huyghens formulated and used the idea that forms the groundwork of the law of conservation of energy in the department of mechanics proper — the principle that the "centre of gravity" cannot rise higher than its original level, if weight alone acts.

Notice, further, that observations with the pendulum played a leading part in the gradual separation of the concepts mass and weight; and in the hands of Newton they were made to furnish more accurate proof than had previously been supplied, that weight is really in constant ratio to

mass at the same place, for all materials. The argument is connected with § 97 above, and deserves to be reproduced here in its main outline.

Pendulums in all respects geometrically alike, and constructed of homogeneous material, are found experimentally to have equal times of beat at the same place for axes similarly situated. Hence, if M and M' are the total weight-moments per unit angle of displacement for two such pendulums (cf. eq. (1(a)), § 97, but writing M for $m_1 g\bar{r}$ to avoid premature assumptions),

$$\frac{I'_O}{I_O} = \frac{M'}{M}. \tag{1}$$

Under the conditions assumed here

$$\frac{I'_O}{I_O} = \frac{m'}{m}, \tag{2}$$

since moment of inertia varies as *density* for the same geometrical distribution. Consequently,

$$\frac{M'}{M} = \frac{m'}{m}; \tag{3}$$

and since the distribution of mass is the same in both bodies on account of their geometrical equality and homogeneousness, the "intensity factor" must be the same; this is g.

Harmonic vibration has weighty affiliations with the problems of molecular or ethereal dynamics involved in all types of wave-motion. The relation sometimes known as "Hooke's Law," announced by him (1678) in the formula, "Ut tensio, sic vis," is followed with sufficient accuracy by springs and elastic materials generally (within ranges that differ for the various cases) to allow us to consider vibrations determined by them as harmonic. They have supplied the visible model according to which the imagination has built up schemes of invisible "elastic motions." But it should be observed that a central force will appear to follow the harmonic law for small disturbances from equilibrium, whatever function the force is of the displacement, if that function can be expanded by using Taylor's Theorem. The circular pendulum is one case in point.

It is an item of recent history, of which we scarcely need to remind ourselves, that we owe to Gauss the establishment of dynamical ideas and measurements within the domain of magnetism and electricity. Such problems as that of § 100 stimulated his mind to plan for their reduction by the methods of Mechanics. Hence followed his invention of so-called "Absolute Measurements" — the C.G.S. system being the one among the first attempts in that direction which has survived.

EXERCISES AND PROBLEMS

1. Express the amplitude of a simple harmonic vibration in terms of the period, and the velocity v at the distance x from the origin.

2. Determine what fraction of the period is required to move from a turning-point through any given fraction of the amplitude, when the motion is simple harmonic.

3. Prove that any oblique projection of a simple harmonic motion is of the same type. What elements are changed by projecting?

4. Change the solution (eq. (2), § 88), into a form that will apply to any epoch angle.

5. Any number of points are moving according to the harmonic law in straight lines, with equal periods, but different amplitudes and phases. The acceleration of a point Q is at each instant the vector sum of their separate accelerations. Determine its motion.

6. Deduce the results in the text for elliptic harmonic motion by considering the ellipse as the projection of a circle.

7. Investigate the result of projecting elliptic harmonic motion upon any pair of conjugate diameters.

8. Prove that the eccentric angle increases at a uniform rate when the motion is elliptic harmonic.

9. A body floating in a stable position is depressed vertically and then released. Under what supposition will the motion be simple harmonic?

10. How will elliptic harmonic motion be modified, if the components parallel to X and Y have a small difference of period?

M

11. Three successive turning-points of a damped harmonic vibration are at points a, b, c, of any scale fixed in the line of motion. If k_1 is small, prove that the position of equilibrium is approximately at the point $\frac{a+c+2b}{4}$.

12. Calculate the fraction of the kinetic energy that is dissipated between two successive passages through the equilibrium position of a damped harmonic motion, the constants being given.

13. Work out the general solution of equation (2), § 91, for all three critical values of $\frac{k_1^2}{k}$, and interpret the integration constants.

14. The rotation-axis of a weight pendulum is not horizontal; how is the period affected?

15. In how far can the plan of treatment in § 100 be paralleled for the weight pendulum?

16. Discuss the limiting cases ($t = \infty$) for the physical and the simple weight pendulum; and show why they differ.

17. In the diagram (Fig. 22), draw an ordinate at each point of OO_1, perpendicular to that line, and proportional to the period for an axis through the point and parallel to Z. What is the locus of the extremities of these ordinates?

18. Calculate in a given case the percentage error introduced by considering a ball hung up by a wire of negligible mass as a simple pendulum.

19. Deduce an approximate expression for the vertical acceleration in a bifilar pendulum, when the amplitude is small, and the ratio $\frac{h}{l}$ large.

20. Prove that the approximation ($\sin\gamma = \gamma$) in any simple circular pendulum effectively substitutes for the circle the cycloid of which it is the osculating circle at the vertex.

21. The four types, (1) torsion, (2) weight, (3) bifilar, and (4) magnet pendulums, treated separately in the text, will occur combined if the physical conditions are superposed. Discuss the combinations (1), (2); (1), (4); (2), (4); (3), (4).

22. What tendencies would show themselves in the types of pendulum treated, if the symmetry supposed in the text did not exist?

CHAPTER VII

THE LAW OF INVERSE SQUARE AND POTENTIAL

102. One main object in the present chapter is to work out some consequences of the law usually associated with Newton's name, the law of gravitation; which we find thus worded: "Between every pair of particles there is a stress of the nature of a tension, proportional to the product of the masses of the particles divided by the square of their distance [apart]." * The law postulates groups of differential forces, and its claim to validity rests upon the agreement of their computed resultants with totals of force and force-moment that are accessible to physical measurement. Satisfactory concordance is found in detail on comparing the phenomena presented within the solar system with deductions from this law. We are obliged to concede that its extension to the "Universe" involves an element of speculation.

In one or two selected instances, we shall need to calculate the gravitational resultants; and in so doing the procedure will be by a process of integration applied to a differential statement. Let u be the distance between any differential masses dm_1 and dm, measured positively from the former; and θ a constant positive magnitude. Then the postulated gravitational force due to the presence of dm_1 and acting upon dm is

$$dU = -\theta \frac{dm_1 \, dm}{u^2}. \tag{1}$$

* Attributed by Tait ("Dynamics of a Particle," ed. 1878, p. 136) to Maxwell, "Matter and Motion"; in which work, however, the phraseology (ed. 1876, p. 114) is not exactly as here cited.

This is open to interpretation as an influence proceeding from dm_1 as a source, and radiated with uniform distribution among all directions. If we define the intensity of that influence at any place as the *force per unit mass* exerted there, it is measured by the coefficient $\frac{dU}{dm}$, which is plainly of the nature of an acceleration having for direction the negative sense of u. The values of the intensity are equal on spherical surfaces described about dm_1 as centre, and from one such concentric sphere to another, this value decreases inversely as the squares of their radii. The product $\theta\frac{dm_1}{u^2}$ being of the order acceleration, it is obvious that θ is not a numerical factor merely, but involves dimensional elements also (cf. § 142–3); its value according to recent determinations is close to 666×10^{-10} in the C.G.S. system (§ 103, end).

Consider now a body of mass m_1 that may be thought of definitely as rigid and filling its volume continuously. The general problem is to find the direction and intensity of the gravitational influence at any point in the region around m_1, due to the combined action of all its parts. This vector quantity shall be called the **gravitation field** of m_1 at the point in question; denote it by G. It is immaterial whether a body is in motion or not; this field accompanies it, retains the same distribution relative to it, and is therefore everywhere expressible as a single-valued point function for any reference-system fixed relatively to the body. The product of dm by the value of G at its position gives the corresponding resultant force due to the whole of m_1.

103. Let it be required to calculate the gravitation field of a sphere composed of homogeneous material. This case has special bearing upon physical and astronomical problems, including the motion of bodies near the earth's surface, and

that of the earth itself. Assume any point P (Fig. 26) at a distance x from the centre C of a sphere with radius r_1 and uniform density δ. The symmetry is such that the direction of the field must be PC; its intensity arises by superposition of effects in that line due to concentric spherical shells. Take any such shell of radius r and thickness dr; intersect it by planes (shown at AB and DE) perpendicular to PC, the angles DCP and ACD being β and $d\beta$ respectively; and calculate first the intensity at P arising from the zonal band of the shell included between the planes. Let u be the common distance of P from elements of the zone, and α the constant angle (u, x). The mass of the band is

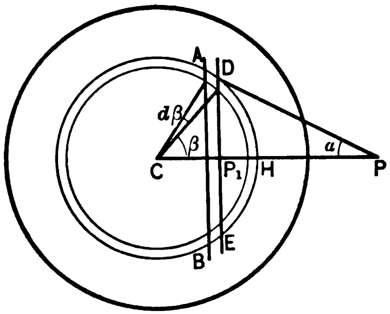


FIG. 26

$$\delta \cdot r\,d\beta \cdot 2\,\pi r \sin\beta \cdot dr = \delta\, 2\,\pi r^2 dr \cdot \sin\beta\, d\beta = \tfrac{1}{2}\,\mu \sin\beta\, d\beta,$$

μ being the mass of the shell. The intensity at P caused by the band is

$$\frac{\theta}{2}\,\mu \sin\beta\, d\beta \cdot \frac{1}{u^2} \cdot \cos\alpha.$$

The whole effect of the shell will be taken account of if the integration variable is changed to u, and the integral is formed between the limits $x - r$ and $x + r$. We have the relations

$$\cos\alpha = \frac{x - r\cos\beta}{u} \quad (1); \qquad u^2 = r^2 + x^2 - 2\,rx\cos\beta. \quad (2)$$

From (1) and (2) we find

$$\cos\alpha = \frac{u^2 - r^2 + x^2}{2\,ux}. \quad (3)$$

Differentiating (2) we obtain

$$\sin\beta\, d\beta = \frac{u\,du}{rx}. \tag{4}$$

Substituting the values from (3) and (4), the previous expression for intensity becomes

$$\frac{\theta}{2}\mu \cdot \frac{u\,du}{rx} \cdot \frac{1}{u^2} \cdot \frac{u^2 - r^2 + x^2}{2\,ux} = \frac{\theta\mu}{4\,rx^2} \cdot \frac{u^2 - r^2 + x^2}{u^2}\,du. \tag{5}$$

The field at P of the entire shell is

$$-\frac{\theta\mu}{4\,rx^2}\int_{x-r}^{x+r} \frac{u^2 - r^2 + x^2}{u^2}\,du = -\frac{\theta\mu}{x^2} \text{ in the line } x. \tag{6}$$

The form of expression (6) shows at once that the field is the same as though the entire mass of the shell were at its centre C. This holds true for the system of concentric shells; consequently the field outside a homogeneous sphere of mass m_1 at any distance x measured from its centre is

$$G = -\frac{\theta m_1}{x^2} \text{ in the line of } x. \tag{7}$$

The point P may approach the surface of the sphere until $x = r_1$. The value of G just outside the surface becomes

$$G_0 = -\frac{\theta m_1}{r_1^{\,2}} = -\frac{\theta\,\frac{4}{3}\,\pi r_1^{\,3}\delta}{r_1^{\,2}} = -\frac{4\,\pi}{3}\,\theta r_1\delta. \tag{8}$$

We shall now return to the expression (6) and consider the conditions when P is inside a hollow sphere. Let the radius of the hollow be CH (Fig. 26). Then at any interior position such as P_1, there are elements of field with opposite sign arising from parts of the same shell to the right and left of it. This situation can be met by changing sign in the lower limit of the integral. We then obtain for the field at P_1 of the shell with radius $r > \overline{CH}$

$$-\frac{\theta\mu}{4\,rx^2}\int_{r-x}^{r+x} \frac{u^2 - r^2 + x^2}{u^2}\,du = 0. \tag{9}$$

Here again the result applies to all the concentric shells comprised in the solid part. Hence throughout such a (concentric) hollow space inside a sphere $G=0$. This is still true though the density varies from layer to layer, provided always that δ is a function of r alone. If then a space is bounded externally by a spherical shell of any thickness (the density being constant at the same distance from the centre); and internally by a concentric (homogeneous) sphere of radius r_2; the field within the space is entirely due to the latter body, and is $-\frac{4\pi}{3}\theta r_2\delta$ at its surface. But the space might be obliterated by increasing r_2, and we obtain thus at the limit *the field within the mass of a solid sphere.* The final use of field being to specify force acting upon a differential mass at a certain position, we may interpret the italicized words to mean that the attractive forces of the rest of a sphere upon an element dm in it at any distance r from the centre, have for resultant a force $-\frac{4}{3}\pi\theta r\delta\, dm$, the solid sphere being now supposed homogeneous. Such elements constitute what are sometimes called the "self-attractive" forces; they may produce internal stresses of great magnitude.

It is a corollary which follows at once from equation (7) above, that the resultant of the mutual attractions between all parts of two homogeneous spheres with masses m_1 and m_2 is accurately a force in the line joining their centres, and equal to $-\frac{\theta m_1 m_2}{x^2}$ when those points are at a distance x apart.

The basis for many calculations in astronomy is thus furnished, because the earth and other planets are sufficiently spherical and homogeneous to justify the assumption of those qualities in them as a first approximation. The simplicity of the integrated law for spheres offers a favorable opening for experimental attack in determining the numerical value of θ. This is so small that comparatively large bodies have been needed in order that the resultant of the attractions should be measurable with accuracy. The simple law for spheres allows the masses to be large, without intro-

ducing laborious corrections for their dimensions. The early and classical experiment is that of Cavendish (1798); it has been repeated with all the refinements of modern accuracy in the torsion-balance, especially by Boys.

104. The consequences of gravitation are traced in phenomena outside the range of terrestrial conditions; and its formal "law of inverse square" is firmly intrenched as an ascertained rule of physical relation. When such attractions are to be included among the forces active, the adoption of a reference-system is legitimate in proportion as it yields accelerations for the masses in question that are accordant with the corresponding elements of gravitation field. That is to say, when gravitation and such other forms of physical action as are known to be affecting any given masses are combined, the resultant accelerations everywhere, calculated from these dynamical data, must be identical with those kinematically determined from the paths and velocities relative to the reference-system used (cf. § 52). It is vital in the conception of our "fundamental system" that we assert this necessary agreement has thus far been found for it. Accelerations relative to any "earth-system" give usually an adequate view of physical action locally upon the earth's surface. We are called upon at this point to ask in how far that system represents the effects of gravitation without distorting them.

There is no physical evidence of forces exerted between the bodies of the solar system and masses outside its limits. The gravitation that we are to take account of in the descriptions of motion here contemplated is therefore composed of stresses internal to that system; and these cannot affect the motion of its centre of mass. On these grounds we are justified in treating that point as having no detectable acceleration, and in choosing it as origin for fundamental reference. The accompanying axes are permanent in direction relative to the general arrangement of (fixed) stars. No change in acceleration is

involved on changing reference to any other elements fixed relatively to these (§§ 34, 36); we shall denote any such system by (F). The earth-system used will be referred to as (E); and as an intermediate step in transferring reference from (E) to (F), we shall find a third system (F_1) convenient, having its axis-directions in common with (F), and its origin with (E), or the corresponding system in similar cases.

105. Suppose the earth and the other body whose motion is before us for description to be homogeneous spheres of mass m_1 and m respectively; and eliminate from the discussion all motion of the earth relative to (F) except translation with its centre C_1, and uniform rotation about its polar axis at the rate $\omega_1 = \frac{2\pi}{86164}$, where the denominator is the number of mean solar seconds in a sidereal day. The conditions shall be further simplified mathematically by making them those of "free fall" in the plane of the equator — no forces but gravitation supposed to act. It has been proved that the resultant then passes through C, the centre of the spherical mass m; hence angular velocity about any diameter will be unaffected, and the motion of C alone needs to be followed. For (F) let the X axis have the direction of a radius drawn outward in the equatorial plane of the earth, and Z that of its northward polar axis. Let the origin for (E) be at C_1, and the axes X_1, Y_1, Z_1, be parallel to those for (F) at the epoch $t = 0$. Such particular choices will not limit the conclusions drawn, when we come to compare the accelerations with respect to (F) and (E) as physical quantities.

The difference between the accelerations of C referred to (F_1) and (F) is the acceleration of C_1 relative to the latter system (§ 34); which can arise under the limitations imposed only from the gravitation field at C_1 of other bodies comprised within the solar system. The field at that point due to m is

too minute for physical recognition when that mass is any movable object near the earth's surface; the fields of the sun and moon are of magnitude that justifies their inclusion in corrective terms under certain conditions, but their ratio to the earth's field at C is small. We shall therefore reserve as corrections the influence of other bodies than m and m_1 upon each of these masses, and for the first approximation consider the acceleration of C in terms of (F_1) as central, its magnitude being $G = \frac{\theta m_1}{r^2}$, if r is the distance C_1C.

The change of intensity in the field G connected with change in r can be written

$$dG = \frac{2\,\theta m_1}{r^3}\,dr; \tag{1}$$

giving

$$\frac{dG}{G} = \frac{2\,dr}{r} = \frac{2\,h}{r} \text{ approximately,} \tag{2}$$

when the radial distance h is a small fraction of r. For fall through 3×10^4 cm. to the earth's surface we have

$$\frac{dG}{G} = \frac{6 \times 10^4}{6.37 \times 10^8} = 9 \times 10^{-5}.$$

This is within the limits of experimental error, unless extraordinary precautions are taken to secure accuracy. We shall use G as a constant magnitude.

The acceleration being central, the condition $r^2\omega = r_0^{\,2}\omega_0$ is fulfilled, where ω is the angular velocity of r (§ 90). Hence, if the mass m originally partakes of the earth's rotation at a height h, and afterward falls freely to the earth's surface under the influence of the field G,

$$\omega = \frac{(R+h)^2}{R^2}\,\omega_1 = \left(1 + \frac{2\,h}{R}\right)\omega_1, \tag{3}$$

for a small ratio of h to the earth's radius R. The same order of approximation as that used above gives $\omega = \omega_1$ and constant.

Assume C at rest relatively to (E) at $t=0$, with coördinates $z_0 = y_0 = 0$, $x_0 = R + h$. Then the corresponding initial conditions for (F_1) show the same coördinates, and also $v_z = v_x = 0$; but $v_y = \omega_1 x_0$. With a central acceleration of constant magnitude G we have, referring now to (F_1),

$$-G = \frac{dv_r}{dt} - r\omega^2; \tag{4}$$

while $\omega = \omega_1$ to the first order of small quantities. In terms of (E) the expressions for the same motion are

$$\frac{d^2x_1}{dt^2} = \frac{dv_r}{dt} = -(G - r\omega_1^2); \quad \frac{d^2y_1}{dt^2} = 0. \tag{5}$$

The latter is the first approximation corresponding to $\omega = \text{const.}$ The (apparent) force $-m(G - r\omega_1^2)$ is directly given by experience as the **weight** of the mass m. The order of accuracy that rates $\omega = \omega_1$ involves neglecting the change of r between $R+h$ and R; hence to the extent that G is constant the difference $(G - r\omega_1^2)$ is also. We know this quantity under the name, "acceleration of gravity"; it is the local weight-constant, and the letter g is almost universally set apart to denote it (cf. § 52, end). The value of G at the equator and sea-level is $981.5 \dfrac{\text{cm.}}{\text{sec.}^2}$; and that of $r\omega_1^2$ is $3.4 \dfrac{\text{cm.}}{\text{sec.}^2}$; an element by no means negligible when we are concerned with *gravitation* and not *weight*.

106. The results thus far obtained for the equatorial plane can all be easily extended to any other locality on the earth at an angular distance λ_1 from the equator. There is then acceleration parallel to Z for both (E) and (F_1),

$$\frac{d^2z}{dt^2} = -G \sin \lambda_1. \tag{1}$$

The motion projected into the XY plane proceeds as before, with $G \cos \lambda_1$ substituted for G, and a radius-vector $r_2 = r \cos \lambda_1$,

r being still the distance of C from C_1. The weight constant g is always the acceleration of C for the system (E). Consequently the general relation that appears is

$$\widehat{G} = \widehat{g} + \widehat{r_2\omega_1^2}. \tag{2}$$

The *directions* of G and g no longer coincide; that of the former is radial to the earth, the latter determines the **vertical**. Latitude λ, which depends upon the angle between the vertical and the polar axis, is thus not in general exactly equal to λ_1; but $\widehat{G} = \widehat{g}$ at the poles.

All of the foregoing discussion is subject to the condition that m is very small in comparison with m_1. This renders it possible to eliminate acceleration of the earth's centre produced by the former mass. It may be well to point out in this connection, however, that phenomena on the astronomical scale present noticeable values for the ratio $\dfrac{m}{m_1}$. For instance, it is 0.011 in the case of earth and moon. In order to bring out specially the result under such conditions, let us consider two homogeneous spheres with masses m and m', and centres at C and C', isolated from all influences but their mutual attractions. Assume as before a system (F_1) with origin at C'; and let it be required to express the relation between the accelerations of C referred to (F_1) and (F). Measure r from C' to C. Then we have in terms of (F), p_r and p'_r being central field or acceleration at C and C' respectively,

$$m'p'_r + mp_r = 0. \tag{3}$$

In terms of (F_1), the radial acceleration of C for pole at C' is (§ 34)

$$\frac{dv_r}{dt} - r\omega^2 = -p'_r + p_r = \left(\frac{m}{m'}p_r + p_r\right). \tag{4}$$

From this we obtain

$$\frac{dv_r}{dt} - r\omega^2 = \frac{m' + m}{m'}p_r. \tag{5}$$

The (apparent) force at C, inferred from its acceleration in terms of (F_1), is larger than the (real) force of attraction in the ratio $\frac{m'+m}{m'}$. When m is negligible as compared with m' this ratio becomes unity, and the case already discussed reappears.

107. We pass now to the second approximation for ω (§ 105), and the evaluation of other corrective terms. Write $\omega=\omega_1+\omega_2$, ω_2 being a variable complement to the constant ω_1. Then a radius-vector r turning at the rate ω will diverge at the rate ω_2 from r_1, if the latter turns with angular velocity ω_1. Calling γ_2 the angular divergence between r and r_1, differentiation gives

$$\frac{d^2\gamma_2}{dt^2}=\frac{d\omega}{dt}=-\frac{2\,\omega v_r}{r}. \tag{1}$$

It is legitimate to use in equation (1) the conditions of the first approximation; that is, ω and r constant, with $v_r=-gt$. Integrating (1) on this basis, we find for $\gamma_2=0$, $\omega_2=0$, at $t=0$,

$$\gamma_2=\tfrac{1}{3}\,\frac{\omega}{r}\,gt^3. \tag{2}$$

At a distance R from C_1 the arc included between r and r_1 is

$$R\gamma_2=\tfrac{1}{3}\,\omega_1 gt^3. \tag{3}$$

This is very approximately the gain at the earth's surface of C as compared with the radius which contained its original position, and which has now turned through an angle $\omega_1 t$. This "eastward deviation" can be observed in a body let fall from a considerable height. For a locality at latitude λ, the deviation is $\frac{1}{3}\,\omega_1 gt^3\cos\lambda$; the factor $\cos\lambda$ projects gt into the plane of the equator.

The gravitation field of the sun at the centre of the earth, if measured by the average central acceleration of C_1, with origin

at the sun's centre and axes those for (F), is $\left(\frac{2\pi}{T}\right)^2 r$. Here T, the period of the earth in its orbit, is 3.156×10^7 seconds, and r, the mean distance between centres, is 1.49×10^{13} cm. We find, then,

$$G_s = -\left(\frac{2\pi}{3.156}\right)^2 \times 1.49 \times 10^{-1} = -0.59 \frac{\text{cm.}}{\text{sec.}^2}. \tag{4}$$

Regard the moon's field as central; in order to express it directly from the data, we have the mass of the moon $m_1 = 6.98 \times 10^{25}$ grams, and the mean distance between centres $r = 3.84 \times 10^{10}$ cm. We calculate

$$G_m = -\frac{\theta m_1}{r^2} = \frac{666 \times 6.98}{(3.84)^2} \times 10^{-5} = -0.003 \frac{\text{cm.}}{\text{sec.}^2}. \tag{5}$$

The negative sign in both cases marks the field as directed toward the centre of sun or moon. The resultant field at the centre of the spherical mass m is then a vector sum of such terms, denoting by $\Sigma(\widehat{k})$ all corrections for position,

$$\widehat{G}_1 = \widehat{G} + \widehat{G}_s + \widehat{G}_m + \Sigma(\widehat{k}). \tag{6}$$

The angles between the components are variable, as well as the magnitudes.

It is deserving of remark that the quantitative distinction between weight and the sum of terms in (6) varies in importance with the nature of the physical questions at issue. The greater number of cases involve only forces among bodies at a given locality, or other *differential* effects upon them. These can be measured when accelerations relative to (E) are known, for in the same region the resultant gravitation-field, including that of the earth, may be regarded as common, and cancelled by subtraction. All terms of (6) may not need to be considered, even in comparing intensities of gravitation for two places. Placing $\Sigma(\widehat{k})$ equal zero, we can write

$$G'_1 - G''_1 = [\widehat{G}' + \widehat{G}_s + \widehat{G}_m] - [\widehat{G}'' + \widehat{G}_s + \widehat{G}_m] = G' - G''. \tag{7}$$

This amounts to neglecting variations from G_s and G_m at different positions on the earth. At each place

$$\widehat{G} = \widehat{g} + \widehat{r_2\omega_1^2}.$$

But, on the other hand, note that the circumstances may make these very differential effects prominent and essential, the tides of the ocean being a familiar instance. They are due to differences of intensity in field for three regions, here arranged in order of decreasing magnitude for the intensity: (1) the water on the side nearer the sun or moon; (2) the centre of the solid earth; (3) the water on the side opposite to the sun or moon. The general result is to modify the weight field of the earth and to separate these portions of the system along the line of centres in each case. That one of the attracting bodies produces the greater tide, whose *differential* action is greater upon the three regions. Calculation shows this to be the moon, in conformity with observation. Let D be the distance apart of centres for earth and either sun or moon. Then we have in both cases approximately, for a distance $D \pm R$, G being the field,

$$G = -\frac{\theta m}{D^2}; \quad dG = 2\frac{\theta m}{D^3}dD = \pm 2\,G\frac{R}{D}. \tag{8}$$

For the sun calculation shows

$$dG = 2 \times 0.59 \times \frac{6.37 \times 10^8}{1.49 \times 10^{13}} = 5 \times 10^{-5}\frac{\text{cm.}}{\text{sec.}^2}. \tag{9}$$

For the moon the result is

$$dG = 2 \times 0.003 \times \frac{6.37 \times 10^8}{3.84 \times 10^{10}} = 1 \times 10^{-4}\frac{\text{cm.}}{\text{sec.}^2}. \tag{10}$$

The centre of gravitation for a mass m can be defined as a point at which the field G due to other bodies has such direction and intensity that the product mG determines by its magnitude, direction, and *position*, the resultant of the gravitational forces acting on m. Such a point and centre of mass are therefore defined by different conditions to be fulfilled. They have been proved to coincide accurately when m is a homogeneous sphere in certain fields. The two points will not be sensibly different, so long as it is allowable to regard gravitation (and weight) as forces constant in direction and intensity throughout the region occupied by the mass m. The use of the phrase "centre of gravity" as an alternative for centre of mass has been noted already (§ 85).

The homogeneous sphere also possesses the property that the directions of its resultant field in the region around it intersect in a common point. The term "centrobaric" has been devised to designate this quality in bodies.

108. Within the range of distance necessary in order to include the phenomena of astronomy, the gravitation field of a

given body cannot be regarded as constant in either intensity or direction. We are compelled to recognize this fact when the motion is to be traced of a satellite in its orbit round its planet, or of a planet round the sun. Some general characteristics of such paths (orbits) are next to be developed, for the supposed case of two homogeneous spheres, isolated from the action of all force except their own gravitational stresses.

Let their masses be m and m_1; their centres C and C_1; and denote C_1C by r. The field of m_1 at C is $-\frac{\theta m_1}{r^2}$; and this is also the (physical) acceleration there. For the system (F_1), however, with origin at C_1, the (apparent) acceleration of C is $-\frac{\theta m_1}{r^2}\left(\frac{m+m_1}{m_1}\right)$ (eq. (5), § 106). The "apparent" factor of $\frac{1}{r^2}$ we shall denote by μ; so that in terms of (F_1) we can say the orbit is described with central acceleration $p=-\frac{\mu}{r^2}$. The path described by C (the orbit of the body m) is then determined by a central force; the rate at which the radius-vector passes over the area included within the orbit is therefore constant; call that rate $\frac{c}{2}$. It shall first be proved that the orbit in terms of these elements is a conic section.

The expression for resultant acceleration can be put into the form, the pole being at the centre of force,

$$p=\frac{dv_r}{dt}-r\omega^2. \tag{1}$$

Without particularizing the law of force, equation (1) can be transformed conveniently for integration by introducing as independent variable γ, the polar angle, and writing u for $\frac{1}{r}$. In order to execute the change of variable, we have the relations,

$$r=\frac{1}{u};\quad r^2\omega=\frac{\omega}{u^2}=c. \tag{2}$$

Hence, $$\frac{dr}{dt} = -\frac{1}{u^2}\cdot\frac{du}{d\gamma}\cdot\frac{d\gamma}{dt} = -\frac{1}{u^2}\cdot\frac{du}{d\gamma}\cdot cu^2 = -c\frac{du}{d\gamma}; \tag{3}$$

$$\frac{d^2r}{dt^2} = -c\frac{d}{dt}\left(\frac{du}{d\gamma}\right) = -c\frac{d^2u}{d\gamma^2}\cdot\frac{d\gamma}{dt} = -c^2\frac{d^2u}{d\gamma^2}u^2. \tag{4}$$

Further, $$r\omega^2 = \frac{1}{u}c^2u^4 = c^2u^3. \tag{5}$$

Making these substitutions, we obtain

$$p = -c^2u^2\left(\frac{d^2u}{d\gamma^2} + u\right). \tag{6}$$

But now specially, $p = -\frac{\mu}{r^2}$ for the law of gravitation; hence,

$$\mu u^2 = c^2u^2\left(\frac{d^2u}{d\gamma^2} + u\right); \tag{7}$$

$$\frac{d^2u}{d\gamma^2} = -\left(u - \frac{\mu}{c^2}\right). \tag{8}$$

The equation last written is of harmonic type for the variable

$z = u - \frac{\mu}{c^2}$ as a function of γ; for evidently,

$$\frac{d^2z}{d\gamma^2} = -z. \tag{9}$$

With the conditions $t = 0$, $z = z_0$, $\frac{du}{d\gamma} = z'$, $\gamma = \gamma_0$, the first integral is

$$\frac{dz}{d\gamma} = \pm\sqrt{z'^2 + z_0^2 - z^2}. \tag{10}$$

And the second integration yields

$$\gamma = \pm \text{arc cos}\left(\frac{z}{\sqrt{z'^2 + z_0^2}}\right) \mp \left(\text{arc cos}\left(\frac{z_0}{\sqrt{z'^2 + z_0^2}}\right) - \gamma_0\right). \tag{11}$$

This is of the form, A and δ being constants,

$$z = A\cos(\gamma - \delta). \tag{12}$$

And rewriting in terms of u and r, we find

$$u = \frac{1}{r} = \frac{\mu}{c^2} + A\cos(\gamma - \delta); \tag{13}$$

$$r = \frac{\dfrac{c^2}{\mu}}{1 + A_1 \cos(\gamma - \delta)}. \tag{14}$$

Equation (14) identifies the orbit as a conic section with a focus at the origin. Accordingly we may simplify by using the known properties of conics. So measuring γ from the axis through a vertex, the conditions are $t = 0$, $\gamma = 0$, $u = u_0$, $\dfrac{du}{d\gamma} = 0$; the last expressing the fact that the tangent is then perpendicular to r_0 (cf. eq. (2), § 88). The modified form of equation (11) is

$$\gamma = \pm \text{ arc cos } \frac{z}{z_0}. \tag{15}$$

The upper sign may be taken, in view of the symmetry of conics. We obtain, finally, after simple reduction,

$$r = \frac{\dfrac{c^2}{\mu}}{1 + \left(\dfrac{c^2}{\mu r_0} - 1\right)\cos\gamma}. \tag{16}$$

On comparing with the standard form of polar equation it is seen that $\dfrac{c^2}{\mu}$ is the semi-parameter, and $\left(\dfrac{c^2}{\mu r_0} - 1\right)$ the eccentricity. The semi-major-axis a can be calculated from the relation,

$$\frac{1}{a} = \frac{1 - \left(\dfrac{c^2}{\mu r_0} - 1\right)^2}{\dfrac{c^2}{\mu}} = \frac{2}{r_0} - \frac{c^2}{\mu r_0^2} = \frac{2}{r_0} - \frac{v_0^2}{\mu}. \tag{17}$$

Here $v_0 = r_0\omega_0$; and the plane of the orbit is fixed by r_0 and v_0. The familiar discrimination among conics on the basis of eccentricity shows that the orbit will be elliptic, parabolic, or

hyperbolic according as the quantity $\frac{c^2}{\mu r_0}$ is less than, equal to, or greater than 2. For the connection of these results with a velocity given at *any* point of the orbit, and their interpretation in terms of that velocity, see § 111.

As current technical terms, we find perihelion for the point in an orbit nearest the centre of force; an apse is a point at which the radius-vector from the focus is either a maximum or a minimum.

109. When the orbit is an ellipse the period is $T = \frac{2\pi ab}{c}$, for r sweeps over area at the uniform rate $\frac{c}{2}$. But the semi-parameter $\frac{c^2}{\mu} = \frac{b^2}{a}$; hence,

$$c^2 = \frac{\mu b^2}{a} = \frac{4\pi^2 a^2 b^2}{T^2}; \quad \mu = \frac{4\pi^2}{T^2} a^3. \tag{1}$$

Under different initial conditions, another ellipse described about the same centre of force by a sphere of *equal mass* would give

$$\mu = \frac{4\pi^2}{T_1^2} a_1^3. \tag{2}$$

Combining (2) with (1): $\frac{T_1^2}{T^2} = \frac{a_1^3}{a^3}.$ (3)

Observe that the squares of the periods are directly proportional to the cubes of the major axes. The condition of isochronism is not fulfilled here, a point of difference from simple harmonic motion. The italicized condition above is necessary to secure identity of value in μ, because that quantity depends upon both m and m_1. However, so long as the ratio $\frac{m}{m_1}$ is small, the same comparison of (1) with (2) can be made for different values of m. This supposition is a sufficient first approximation for the sun in its relation to the planets; the mass of the sun is about 750 times that of all the planets.

The preceding results have been presented here as deductions from the (postulated) law of gravitation. But historically the three most important among them were discovered as empirical rules by Kepler, from a discussion of observations, and Newton was led to formulate the one inclusive statement as their common source. The three laws known as Kepler's are:

(1) A radius-vector drawn from the sun's centre to that of a planet describes equal sectors in equal times. (Cf. § 108; this holds true for all central forces.)

(2) The orbits of the planets are ellipses, with the sun's centre at one focus. (Cf. § 108, eq. (16), and note the greater comprehensiveness of its solution.)

(3) The squares of the periodic times of two planets are proportional to the cubes of the major axes of their orbits. (It has just been proved why the more exact law takes this approximate form.)

110. By way of supplement to this part of our subject, one form of converse proof, such as Newton needed to discover, is inserted: the orbit being known, to deduce the law. Granted that the force is central, and the path an ellipse with the sun at one focus; establish the law of force.

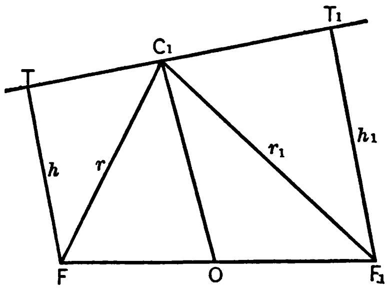


FIG. 27

Let F and F_1 (Fig. 27) be the foci, C_1 the earth's centre, TT_1 the tangent, r and r_1 the radii-vectores, h and h_1 perpendiculars to the tangent. Then the normal acceleration at C_1 is $-\frac{v^2}{\rho}$; it is also $p\frac{h}{r}$ by projection of the resultant (central) acceleration p. Again, the condition $r\omega^2 = c$ is

equivalent to $hv = c$; for $h = r\cos\beta$, and $r\omega = v\cos\beta$. The necessary relation, therefore, in the case of any central orbit is

$$p = -\frac{c^2 r}{h^3 \rho},$$

and the acceleration is *independent of the law thus far.*

Specifically, introducing elementary properties of the ellipse, we have

$$\rho = \frac{(r_1 r)^{\frac{3}{2}}}{ab};\ hh_1 = b^2;\ r_1 r = b'^2 \text{ for } OC_1 = a';\ hr_1 = rh_1.$$

Multiply the last equation by hr, use the two preceding relations, and we obtain $h^2 b'^2 = r^2 b^2$. Hence $h^3 = r^3 \frac{b^3}{b'^3}$; and further, $\rho = \frac{b'^3}{ab}$. Consequently,

$$p = -c^2 r \cdot \frac{ab}{b'^3} \cdot \frac{b'^3}{r^3 b^3} = -\frac{c^2 a}{b^2} \cdot \frac{1}{r^2}. \tag{1}$$

The semi-parameter is $\frac{b^2}{a}$; and comparing with equation (16), § 108, we find $\mu = \frac{c^2 a}{b^2}$, which leads to the conclusion,

$$p = -\frac{\mu}{r^2}. \tag{2}$$

By inference from Kepler's third law, μ is constant for all the planets.

111. The formula for central acceleration used in § 108 is $p = -\frac{\mu}{r^2}$. The corresponding force acting upon m at C being $mp = -\frac{\mu}{r^2} m$, the work of this force during displacement through an element ds of the orbit can be written, denoting the angle (v, r) by β,

$$dW = -\frac{\mu}{r^2} m \cos\beta\, ds. \tag{1}$$

But $\cos\beta\, ds = dr$; and substituting this value we find by a simple integration the work for the interval r_0 to r:

$$W = \mu m\left(\frac{1}{r} - \frac{1}{r_0}\right). \tag{2}$$

Let v_0 and v be the speeds at r_0 and r; then the work equation is

$$\mu m\left(\frac{1}{r} - \frac{1}{r_0}\right) = \frac{m}{2}(v^2 - v_0^2); \tag{3}$$

or

$$v^2 = v_0^2 + \mu\left(\frac{2}{r} - \frac{2}{r_0}\right). \tag{4}$$

For the same centre of force this value of v depends entirely upon the numerical magnitude of v_0, with the terminal values of r. It is not a function of the particular curve which may be the orbit, nor of the mass m. Let any spheres be described about C_1, and let the speeds be equal in any orbits, where they cut one of the spheres. Then for all of those orbits that reach any other sphere the speeds will also be equal at the points where they intersect it. Of course such central orbits must lie each in some plane through C_1.

From equation (4) it can be seen that the quantity $\left(v^2 - \frac{2\mu}{r}\right)$ remains constant during the motion in any orbit, and equal to its value $\left(v_0^2 - \frac{2\mu}{r_0}\right)$ at any arbitrarily chosen instant. This constant is determined (for the same centre of force) by r_0 and the speed v_0; hence it does not depend upon the *direction* of the initial velocity. By comparison of (4) with equation (17), § 108, the relation appears:

$$\frac{2\mu}{r} - v^2 = \frac{2\mu}{r_0} - v_0^2 = \frac{\mu}{a}. \tag{5}$$

Consequently the length of the major axis (and of the period also, eq. (2), § 109) is the same for all orbits described about

the same centre, in which the speeds are equal at any given distance.

Equation (5) furnishes the criterion for the type of conic in another form as depending upon this quantity $\left(\frac{2\mu}{r} - v^2\right)$. The orbit is elliptical, parabolic, or hyperbolic, as this difference is positive, zero, or negative. It has just been shown that it retains its value all round a given orbit. The above result agrees with that of § 108, for remembering that v_0 and r_0 are there applied to an apse, we have

$$e = \frac{c^2}{\mu r_0} - 1 = \frac{v_0^2 r_0}{\mu} - 1; \quad \text{and for } e \lesseqgtr 1, \quad v_0^2 \lesseqgtr \frac{2\mu}{r_0}.$$

The critical comparison is then to be made between the speed v at any position, and the quantity $\frac{2\mu}{r}$ depending upon the centre and the distance from it. The dynamical significance of the criterion becomes clear when we multiply by $\frac{m}{2}$. Then $\frac{mv^2}{2}$ is the actual kinetic energy, and $-\frac{\mu m}{r}$ is the work that would be done by the field forces if m moved from the position r and *just came to rest* at "$r = \infty$"; since it is the limit of equation (3) for those values. This is, of course, only a form of answer to the question: "Are the field forces strong enough to exhaust the kinetic energy?" If they are (within a finite distance), the path will be a closed curve; if not, it will have an infinite element.

112. The result announced in conformity with equation (3) of the preceding section is there connected with a special instance, but its essential feature repeats itself persistently under varied physical conditions. Systematic knowledge of the circumstances in which work is a function of position alone for a given body becomes therefore of necessity a prominent part of our science. Several important sources of force

in nature exhibit the characteristic in common, that they appear to radiate their influences by some mechanism of a field. The law of intensity in such fields is that of inverse square (when differentially stated, as explained in § 102), provided always that they are "steady," *i.e.* not subject to fluctuations with time. In many actual combinations, then, the amounts of work to be dealt with depend upon the relative positions alone — the "configuration" — of the bodies concerned.

The expression of the total field stress between two bodies will in general call for two integrations. One of these determines the resultant field at any point; by means of the other we calculate the resultant (force and force-moment) thereby exerted upon a body of finite dimensions placed in the field.

The energy relations under such general conditions will be exemplified somewhat further by means of the gravitation field, as an aid in acquiring command of the necessary conceptions. In order to present these in simplest form, we shall continue to accept the restrictions of § 108 that confine the discussion to the case of two homogeneous spheres, or one sphere and a body regarded as a particle at its own centre of mass. By observing the indicated limitations, difficulties that are merely mathematical can be postponed, while an instructive preliminary view is gained.

Let the notation be that of § 108. The work W of the force $mG = -\frac{\theta m_1}{r^2} m$ can be expressed by simple integration for displacement along any path terminating upon spheres of radii r_0 and r, giving

$$W = \theta m_1 m \left(\frac{1}{r} - \frac{1}{r_0}\right). \qquad (1)$$

The work of the gravitational stress here represented obviously involves the total contraction of the distance between centres,

and equation (1) recognizes that fact in the use of the coördinate r. But work is done at each end of the stress line, since both bodies move in it (if either does), the effects being divided in the inverse ratio of the masses m_1 and m. For the forces acting upon them are equal in magnitude (two aspects of the same resultant stress), and the displacements relative to the system (F) are in the inverse ratio of the masses (as a consequence of the *constant* ratio of their accelerations, cf. § 131).

113. Under the conditions here assumed to prevail, the work of the field forces is expressed in terms of r as a single-valued point function. When the path of C relative to C_1 as origin begins and ends at the same value of that coördinate, the work is zero; as a particular case the path may be a closed curve. If the path is such that r_0 and r are not equal, kinetic energy appears in m_1 and m, or disappears from them. There is one valuable advantage accruing from the simple terms in which work is here expressible, and that is the possibility of assigning its value beforehand to the work that must inexorably be done by the field forces while any given displacement is occurring. But all the work of forces acting upon a given system up to the end of a definite interval must necessarily be included in the sum of what has already been done at any epoch, and what still remains to be done. If we regard the first element as given, for a mass m and the field forces, in terms of the kinetic energy at any position r_0, the second may be covered by equation (1), § 112, applying to the same position, if the interval contemplated ends at the value r. Evidently a similar division of work might be indicated in any instance whatever, but the plan is not feasible unless we can give simple expression at any instant to the work still to be done, and it is practically limited to field forces.

Since all these relations for field are independent of the choice of any particular path, a displacement from r_0 to any other position r can be conceived as part of a path extending from $r = r_0$ to $r = \infty$, and comprehending all the possibilities of the future. In this conventionalized sense, the work of the field forces from $r = r_0$ to $r = \infty$ is the entire work remaining to be done for the interval $t = t_0$ to $t = \infty$. This quantity assumes a peculiar importance from the present point of view, because the difference between two such values is the work $[W]_{r_0}^{r}$ corresponding to any other interval $r = r_0$ to $r = r$. For these are definite integrals, and we can write

$$[W]_{r_0}^{r} = [W]_{r_0}^{\infty} - [W]_{r}^{\infty}. \tag{1}$$

This is only equation (1), § 112, in new symbols; for we see that

$$[W]_{r_0}^{\infty} = -\theta m_1 m \cdot \frac{1}{r_0}; \text{ and } [W]_{r}^{\infty} = -\theta m_1 m \cdot \frac{1}{r}. \tag{2}$$

Observe the order of terms in any such difference as that expressed in equation (1).

It has been noted in defining kinetic energy (§ 59) that the word "energy" means work or its measured equivalent. Work of field forces expressed according to the type of equations (2) is known as **potential energy** at the position r_0 or r, the qualifying adjective distinguishing this quantity from kinetic energy at the same position. As a formal statement, "The potential energy of a mass m at any position in a gravitation field is the work that would be done upon it by the field forces during a displacement from that position to an infinite distance."

The sum of the kinetic and the potential energy at any epoch represents the total endowment with energy at that time. It contains the realized value of the present in the kinetic energy; and under potential energy are comprised the possibilities of the indefinite future — according to the interpre-

tation just given. Equation (1) exhibits the work for any actual interval as the difference of potential energy at its terminal points. It results from the conventions as to sign that a negative change in potential energy goes with a positive change in kinetic energy. Denoting the two energy quantities by $E(P)$ and $E(K)$, we may then write the energy equation,

$$-[E(P)-E_0(P)]=E(K)-E_0(K). \qquad (3)$$

From which follows,

$$E(P)+E(K)=E_0(P)+E_0(K). \qquad (4)$$

The constancy of the energy (including both its terms) is a special example under the comprehensive physical law of conservation of energy. Such a system as we have been considering is a **conservative system.**

The result now obtained furnishes a new interpretation for equations (4) and (5), § 111; for the quantity $\left(v^2-\frac{2\mu}{r}\right)$, when multiplied by $\frac{m}{2}$, is the sum of the kinetic and the potential energy ($\mu=\theta m_1$). Whether the mass m will remain within the field or move out of it is determined by the sign of that sum.

The use of the term "potential energy" as connected with weight in elementary physics corresponds really to the difference of potential energy between two levels, applied to the (apparent) forces of the earth's weight-field. This elementary usage is equivalent to transferring the standard position from infinity to the *lower* level, which makes potential energy at the upper level positive. The current conventions as laid down in the text are chosen with reference to greatest simplicity in connection with fields of magnets and electrified bodies, as well as gravitation.

The view is probably faulty that regards this type of potential energy as stored up in the body exposed to the field. The physical conception of the situation tends rather toward accepting as truth a transfer of energy to the field medium, when bodies are separated against the resistance of their attraction. Where work is done in producing molecular strain (as in a distorted spring) there seems to be no reason why the energy should not be associated in thought with the body.

It serves a certain purpose to speak of potential energy as representing work that has been done, in cases of repeated oscillatory motion, or of bodies known to have been lifted from the earth's surface. The work represented, however, was not done *by* the field forces, but *against* them.

114. In order to put the idea of potential energy in a form ready for application to any mass in the field (any differential mass in the general case), it is usual to express the work per unit mass, $\frac{W}{m}$ or $\frac{dW}{dm}$, for a path from $r = r$ to $r = \infty$, instead of the work itself. The symbol V has been appropriated to this conception of "work intensity"; and it is called **potential**. For the case under discussion we have, using a subscript to denote the position to which the value applies,

$$V_r = -\theta m_1 \cdot \frac{1}{r}. \tag{1}$$

Difference of potential energy, also reckoned per unit mass, appears as **difference of potential**. Surfaces on which the potential has a constant value are then suggestively named equipotential surfaces; they are spheres with C_1 as centre in the present instance.

From what has preceded it is clear that the relation between gravitation field and potential is

$$G_r = -\frac{dV_r}{dr}. \tag{2}$$

Similarly the central force at any position is the derivative with respect to r, taken negatively, of the potential energy at that position.

It has been insisted upon already that, under the conditions assumed, work depends upon what we have called configuration only, and requires no data respecting rest or motion; we see, too, that potential energy involves no coördinate but r. Kinetic energy enters, however, as another aspect of the energy relations; and in order to assign values to this quantity, a

reference-system must be introduced. This is, speaking accurately, always (F); but in most cases (F_1) may be substituted, or even (E) where we do not need to go further than the (apparent) forces of the earth-system. If the latter is used, the weight-field g replaces G. Even for such important examples as the sun's influence upon the planets we can place the ratio $\frac{m}{m_1}=0$ as a first approximation; so we shall assume a system (F_1) with origin at C_1. Let XYZ be the axes, G the field. The work is the sum of the parts done by the three projected forces. Consequently (§ 58, eq. (5)),

$$dE(P) = -mGdr = -m\,[G_x dx + G_y dy + G_z dz]; \qquad (3)$$

$$\begin{aligned} dV &= -Gdr = -[G_x dx + G_y dy + G_z dz] \\ &= \left[\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right]. \end{aligned} \qquad (4)$$

Further, $\frac{\partial V}{\partial x} = \frac{dV}{dr} \cdot \frac{\partial r}{\partial x} = -G\cos\alpha = -G_x$, and similarly for Y and Z. We conclude that the field parallel to any (fixed) line is the derivative of the potential with respect to the coördinate in that line, but taken with reversed sign. Any such projected field will give the acceleration in its direction as a basis for further calculation of the kinematical quantities.

The essential condition that no other forces are acting must be kept steadily in view. An instructive case to consider in parallel with the text (although the law of force is different) is that of two spheres with equal radii but unequal mass, joined on the line of centres by a stretched elastic cord, and moving on a smooth horizontal table. Examine particularly the changes in the partition of the kinetic energy as consequences of holding one ball "fixed," or in any way modifying the motion producible by the cord tension alone. Under these circumstances an external force is added, and the centre of mass of the system has acceleration.

EXERCISES AND PROBLEMS

1. Calculate the field of a homogeneous circular disk of differential thickness and of radius r, at a point distant h from its centre in a normal to its plane. What is the limiting value of the field for $\frac{h}{r}=0$?

2. Use the previous result as a step toward calculating the field of a homogeneous cylinder at a point on its axis. Examine limiting values.

3. Express the field due to the joint action of two homogeneous spheres at a point in the line of centres. Distinguish the case where the partial fields are concurrent from that in which they are opposite.

4. The centre of a sphere is upon the prolonged axis of a cylinder, both being homogeneous. Locate the point of zero field in the space between them. Are there impossible cases?

5. Investigate whether the vertical at a given locality on the earth is strictly a straight line, as it is traced upward from the earth's surface.

6. Prove the law of inverse square, assuming Kepler's results, by the use of equation (6), § 108.

7. Discuss the focal ellipse as an orbit, in connection with the circle of "steady" motion. (Compare § 90, end).

8. Prove that earth and moon describe similar orbits about their common centre of mass.

9. Show how to construct an elliptical orbit, when μ is given, with the initial velocity, and the position relative to the focus.

10. Discuss the motion due to its weight, when a particle slides along any smooth fixed guide-curve, as a case of potential energy in the earth's weight-field.

11. Carry out the discussion of potential energy and potential, where the law is that of central attraction proportional to distance from the centre.

12. Account for *all* of the work done by the gravitation stress, when a small mass falls freely near the earth's surface.

13. What criterion does calculus furnish whether the conception of potential energy applies to the force-distribution through a given region?

CHAPTER VIII

OTHER APPLICATIONS OF THE PRINCIPLES

115. The exposition of such principles as our present plan includes is made connectedly in Chapters I–V, in order to emphasize with due weight the sequence and interdependence of the conceptions that are requisite. Nevertheless, it is indispensable that those ideas should be adequately illustrated at every step as the development of them proceeds — else their meaning cannot be interpreted completely. This chapter is given up to a series of problems and minor propositions, so selected as to serve that purpose, and at the same time to exhibit the principles in their relation to physics as well as other sciences in whose interest the study of Mechanics is pursued. The matters treated in Chapters VI and VII contribute to the same general end, the chief reasons for segregating those applications being found in their special importance, and the close connection with each other of the problems in each group. A supplement to this material is added in the General Exercises and Problems.

In seeking the significant features of the solutions that follow, the procedure is necessarily by partial inclusion of dynamical elements. Something has been done under each heading toward indicating how the simplified problem is conceived, especially where misunderstanding appears likely, or the bearing of some particular condition is to be made clear. But it would be cumbersome and pedantic to catalogue completely the limitations in each instance; the statements and

indications given are to be more fully interpreted according to their plainest intention. Explanation of a few conventional usages in description adopted in this chapter, and in the longer list of problems following Chapter X, may find place here.

A **cord** is of negligible mass, flexible, and inextensible unless qualified as elastic. In the latter case strain is proportional to stress.

A **small mass** is one whose greatest dimension is small compared with other lengths involved. The earth may be a small mass for some problems connected with its orbit.

A **homogeneous body** is one whose density is uniform in all its parts. All bodies are rigid unless the contrary is specified or plainly implied.

Sliding contacts are smooth, and guides are fixed, unless the contrary is specially noted.

Finite force described as acting in a line is to be understood either in the sense of § 61, I, or in that of § 80, end.

116. Consider first a body moving vertically with translation and in a vacuum, under the influence of its weight alone, this force being regarded as constant in direction and magnitude. Assume the coördinate s positive downward; then the equation of motion for the centre of mass of m_1 is

$$m_1 g = m_1 \frac{dv}{dt}. \tag{1}$$

On removing the mass-factor a kinematical equation appears, giving the resultant acceleration

$$g = \frac{dv}{dt} = \frac{v\,dv}{ds}. \tag{2}$$

The general application of our results will not be sacrificed by taking for initial conditions $t = 0$, $s = 0$, $v = v_0$; then integration of (2) yields

$$v - v_0 = gt;\ s = \tfrac{1}{2}gt^2 + v_0t;\ \frac{v^2 - v_0^2}{2} = gs. \tag{3}$$

As a special case, if $v_0 = 0$,

$$v = gt;\ s = \tfrac{1}{2}gt^2;\ v^2 = 2\,gs. \tag{4}$$

Again, if $v_0 < 0$, we have for $t_1 = -\frac{v_0}{g}$,

$$v_1 = 0;\ s_1 = -\frac{v_0^2}{2\,g}. \tag{5}$$

For $t > t_1$, $v > 0$, and on return to $s = 0$ we find the values,

$$t = 2\,t_1;\ v = -v_0;\ v^2 = -2\,gs_1. \tag{6}$$

The second equation of (3) can be written

$$s = \frac{v_0 + (v_0 + gt)}{2} \cdot t = \frac{v_0 + v}{2}t. \tag{7}$$

This shows that the average speed for any interval is the mean of the terminal values. The essential condition for this result is the *constant* time-rate of the speed. Restoring the mass-factor in the first and third equation of (3), we obtain the equations of impulse and work (§ 67),

$$m_1(v - v_0) = (m_1g)t;\quad m_1\frac{v^2 - v_0^2}{2} = (m_1g)s. \tag{8}$$

117. The foregoing results are modified when the initial velocity is inclined to the vertical. The motion of a small mass subject to that condition shall be examined. The path must lie in a vertical plane determined by the vectors v_0 and g in connection with the initial position. Let this be XY, with Y vertically upward; assume $x = 0$, $y = 0$, at $t = 0$, and denote the angle (v_0, X) by α_0. Write at once in kinematical form the equations,

$$\frac{dv_y}{dt}=-g;\ \frac{dv_x}{dt}=0;\ v_y=-gt+v_0\sin\alpha_0;\ v_x=v_0\cos\alpha_0;\quad (1)$$

$$y=-\frac{gt^2}{2}+v_0t\sin\alpha_0;\quad x=v_0t\cos\alpha_0.\quad (2)$$

The simultaneous equations (1) and (2) give by elimination of t the equation of the path,

$$y=-\frac{g}{2}\left(\frac{x}{v_0\cos\alpha_0}\right)^2+x\operatorname{tg}\alpha_0.\quad (3)$$

A parabola with axis vertically downward and vertex at (a, b) is represented by $(a-x)^2=2p(b-y)$; and comparison of coefficients with (3) shows for coördinates of the vertex,

$$y_1=\frac{v_0^2\sin^2\alpha_0}{2g};\quad x_1=\frac{v_0^2}{g}\sin\alpha_0\cos\alpha_0=\frac{v_0^2}{2g}\sin 2\alpha_0.\quad (4)$$

At that point, $v_y=0;\quad v_x=v_0\cos\alpha_0;\quad t_1=\frac{v_0\sin\alpha_0}{g}.\quad (5)$

Writing the first two equations of (1) in the alternative forms $\frac{v_y dv_y}{dy}=-g$; $\frac{v_x dv_x}{dx}=0$, and adding their first integrals, we obtain, after multiplying by m_1, the work equation,

$$-(m_1g)y=\frac{m_1}{2}(v^2-v_0^2).\quad (6)$$

Introducing the factors m_1 into the last two equations of (1), the impulse equations appear,

$$-(m_1g)t=m_1(v_y-v_0\sin\alpha);\ 0=m_1(v_x-v_0\cos\alpha).\quad (7)$$

These equations enable us to determine all the elements belonging to the path. For example, the magnitude of v at any position for which either x or y is known is given by (3) and (6); the time by (2); and then the direction of v by $\operatorname{tg}\alpha=\frac{v_y}{v_x}$ from (1).

Because such a parabola is a first approximation to the path of a body thrown or projected, it is called a trajectory. The distance from the origin at which it intersects any surface is its **range** on that surface. The latter is found, of course, by determining values of x and y common to the parabola and the section of the surface by the XY plane.

118. Preserving the general conditions of § 116, let us include among the acting forces a resistance proportional to the speed attained. If k_1 is a positive constant, denoting dynes per unit speed, the equation of motion becomes

$$m_1 g - k_1 v = m_1 \frac{dv}{dt}. \tag{1}$$

This form is valid for both upward and downward velocity, because $k_1 v$ changes sign with v. Divide by m_1, write k for $\frac{k_1}{m_1}$, and we obtain

$$g - kv = \frac{dv}{dt}. \tag{2}$$

Using as initial conditions $t = 0$, $s = 0$, $v = v_0$, the first integral of (2) is

$$t = \frac{1}{k} \log\left(\frac{g - kv_0}{g - kv}\right); \quad v = \frac{g}{k} - \frac{g - kv_0}{k} \cdot e^{-kt}. \tag{3}$$

From equations (3) we conclude that v approaches the limiting value $\frac{g}{k}$ asymptotically. The limit in such cases is known as a **terminal velocity**. Writing $\frac{g}{k} = v_1$, equation (3) for v can be put into the form

$$v = v_0 e^{-kt} + v_1 (1 - e^{-kt}). \tag{4}$$

On account of the exponential factor, there is close approach to the terminal velocity at the end of a moderate time, if k is of noticeable magnitude. A second integration gives

$$s = \frac{g}{k} t - \frac{g - kv_0}{k^2} (1 - e^{-kt}) = v_1 t - \frac{1}{k} (v_1 - v_0)(1 - e^{-kt}). \tag{5}$$

By placing $v = 0$ in (3) or (4), we obtain

$$t_1 = \frac{1}{k}\log\left(\frac{g - kv_0}{g}\right) = \frac{v_1}{g}\log\left(1 - \frac{v_0}{v_1}\right). \tag{6}$$

When $v_0 > 0$, equation (6) is not satisfied for any positive value of t, whatever magnitude k may have; that is, m_1 cannot be brought to rest while moving downward, with the assumed law of resistance. If $g = kv_0$, $t_1 = -\infty$; but notice that this indicated impossibility requires the law of equation (1) to have been always true. As a fact, of course, the mass may have been given such velocity v_0 by a process of projection ending at the epoch $t = 0$, that $v_0 \lesseqqgtr v_1$. The value v_1 is the limit approached in all three cases. If $v_0 < 0$, a positive value of t can be found that fulfils equation (6); and on substituting this in (5), we find as the position of rest from which m_1 begins to move downward,

$$s_1 = \frac{g}{k^2}\log\left(\frac{g - kv_0}{g}\right) + \frac{v_0}{k} = \frac{v_1}{k}\log\left(1 - \frac{v_0}{v_1}\right) + \frac{v_0}{k}. \tag{7}$$

For the motion downward that ensues, it is sometimes convenient to assume as new initial conditions those at the position of rest.

119. A similar plan can be followed where the resistance at any instant is proportional to the square of the speed. Note, however, that since v^2 is always positive, the resistance term does not change sign *automatically* with v. Two equations of motion must therefore be used; that for motion downward takes the form

$$m_1 g - k_1 v^2 = m_1\frac{dv}{dt}; \quad g - kv^2 = \frac{dv}{dt} = \frac{v\,dv}{ds}. \tag{1}$$

Integrate by using the second expression for acceleration, with initial conditions $s = 0$, $v = v_0$, $t = 0$. The result is

$$v^2 = \frac{g}{k} - \frac{g - kv_0^2}{k} e^{-2ks}. \tag{2}$$

Under these conditions, the terminal velocity is $\sqrt{\frac{g}{k}}$; so substituting v_1^2 for $\frac{g}{k}$, equation (2) becomes

$$v^2 = v_0^2 e^{-2ks} + v_1^2 (1 - e^{-2ks}); \quad s = \frac{v_1^2}{2g} \log\left(\frac{v_0^2 - v_1^2}{v^2 - v_1^2}\right). \tag{3}$$

As in the preceding section (for $v_0 > 0$), the value $v = 0$ is excluded for $t > 0$, and the limit, for all values of k and v_0, is v_1, the initial velocity being due to any causes, expressed in equation (1) or otherwise.

By way of preliminary to establishing a second relation, return to equation (1), and write,

$$\frac{dv}{dt} = g - kv^2 = k(v_1^2 - v^2). \tag{4}$$

The integral of this is

$$kt = \frac{1}{2v_1} \log\left[\frac{(v_1 + v)(v_1 - v_0)}{(v_1 - v)(v_1 + v_0)}\right]. \tag{5}$$

From (5) we obtain, denoting the constant factor $\frac{v_1 - v_0}{v_1 + v_0}$ by A,

$$e^{2kv_1t} = \frac{A(v_1 + v)}{v_1 - v}; \quad v = v_1 \frac{e^{2kv_1t} - A}{e^{2kv_1t} + A} = v_1 \frac{e^{kv_1t} - Ae^{-kv_1t}}{e^{kv_1t} + Ae^{-kv_1t}}. \tag{6}$$

If $v_0 = 0$, $A = 1$, and in this case we find

$$v = v_1 \frac{e^{kv_1t} - e^{-kv_1t}}{e^{kv_1t} + e^{-kv_1t}} = v_1 \tanh(kv_1t) = v_1 \tanh\left(\frac{g}{v_1} t\right). \tag{7}$$

When v_0 is negative, we must write for the motion upward, instead of (1), $\frac{v\,dv}{ds} = g + kv^2$; and this leads to

$$v^2 = v_0^2 e^{2ks} - v_1^2 (1 - e^{2ks}); \quad s = \frac{v_1^2}{2g} \log\left(\frac{v^2 + v_1^2}{v_0^2 + v_1^2}\right). \tag{8}$$

And instead of (4), $\frac{dv}{dt} = g + kv^2$; from which follows,

$$t = \frac{1}{kv_1}\left[\operatorname{arc tg}\left(\frac{v}{v_1}\right) - \operatorname{arc tg}\left(\frac{v_0}{v_1}\right)\right] = \frac{v_1}{g}\left[\operatorname{arc tg}\left(\frac{v}{v_1}\right) - \operatorname{arc tg}\left(\frac{v_0}{v_1}\right)\right]. \quad (9)$$

The value $v = 0$ is now possible; and it enters at the position and the time,

$$s_1 = \frac{v_1^2}{2g}\log\left(\frac{v_1^2}{v_0^2 + v_1^2}\right); \quad t_1 = -\frac{v_1}{g}\operatorname{arc tg}\left(\frac{v_0}{v_1}\right). \quad (10)$$

From the position of rest, the motion proceeds in accordance with equation (1). In this discussion the distance is expressed as a function of the time indirectly, by means of two simultaneous equations.

Some forms of frictional resistance to motion in actual cases approximate closely, within certain limits for speed, to one or the other of the values assumed in this section and the one preceding. Other conditions can be covered in part by combinations of resistance proportional to speed and to its second power, using empirical values for the constants. Where a body moves through a "resisting medium," there is usually a leakage of energy into less controllable forms; as regards the body, energy is said to be dissipated. If the medium is a fluid (air, or a liquid like water), the buoyant force must be subtracted from m_1g in the equations of motion.

Changes produced in the parabolic path of § 117, when resistance is proportional to speed, can be dealt with by the method of § 118 applied to each projected velocity, since the term $-kv$ is projected also. But when resistance varies as the square of speed, the modified parabolic motion must be treated by specially devised methods.

120. The types of equation written for weight in § 116 will be found repeated when constant tangential acceleration occurs

under the action of any forces; but s and v must refer to the path, straight or curved. As one example, let the centre of mass of m_1 move along any line of an inclined plane, with tangential force due to weight alone, the guiding surfaces being smooth. If β is the angle (in a vertical plane) between the line and its horizontal projection, equation (2), § 116, becomes $g \sin \beta = \frac{dv}{dt}$, when s is positive down the inclined line. The angle $\beta = i$, the inclination of the plane, if the path is the line of steepest slope; otherwise the relation exists, $\sin \beta = \sin i \cdot \cos \epsilon$, where ϵ is the angle between path and slope-line. This appears in connection with Fig. 28, in which BA is the path, BD a slope-line, and ACD the horizontal projection of ABD. For we have

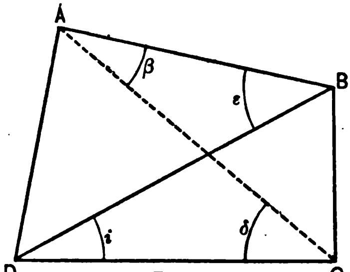


FIG. 28

$$\text{Area }(ABC) \cos \delta = \text{Area }(DBC);$$

$$\overline{BA} \cdot \overline{CA} \cdot \sin \beta \cdot \cos \delta = \overline{BA} \cos \epsilon \cdot \overline{CA} \cos \delta \cdot \sin i.$$

On cancelling common factors the indicated equality results.

The equations of § 116 can now be paralleled, $g \sin \beta$ replacing g everywhere.

Independence of the particular mass-factor is, however, characteristic of all results (for translation or a small mass) when the instantaneous forces are proportioned to the mass that is to be accelerated. For then the ratio of force to mass, which is acceleration, will be the same at corresponding instants for any two masses. The scope here outlined is wider than the instance just brought forward, because the tangential (or resultant) forces may vary in direction and

magnitude without ceasing to fulfil the vital condition. The ratios of force to mass are compared for different masses at corresponding instants, and not for one mass at different instants. If such ratios preserve the same value the equation of motion becomes kinematical, and applies to all masses alike. The suggested extension will find illustration repeatedly as we proceed; we shall confine our attention at present to the motion of a small mass along any smooth curve, with tangential force due to weight only.

The equation of motion,

$$m_1 g \sin\beta = m_1 \frac{dv}{dt}; \quad g \sin\beta = \frac{dv}{dt} = \frac{v\,dv}{ds}, \tag{1}$$

can be made to include the case described by regarding $\frac{\pi}{2} - \beta$ as the variable angle between the vertical and the instantaneous direction of the tangent. The process of integration and the results derived will in general depend upon the nature of the curved path in each instance; but the work equation is an exception, as can be shown without difficulty. Integrate (1) with respect to distance, observing that $ds \sin\beta$ is the vertical projection of any element of the curve, which we shall denote by dx. For the interval (v_0, x_0) to (v, x), the work equation takes the form

$$m_1 g (x - x_0) = m_1 \frac{v^2 - v_0^2}{2}. \tag{2}$$

There is no reference to any particular curve here. This conclusion is also reached by an argument cast in different form, although it rests finally on the same foundation. To say that the curve is smooth signifies that the constraint is exercised in the normal to the path at every point. Consequently its projection upon the tangent is zero. Hence all the work is that of weight, and is the product of the constant force $m_1 g$ and the vertical displacement $(x - x_0)$.

The force normal to the guiding curve is eliminated from consideration by the operation of projecting upon the tangent, in which line the *entire* work is done (§ 58). But the confusion of thought must be avoided that would lead to neglecting the normal force likewise in writing *partial* work-equations. We will put down two such for forces parallel to X and Y, supposing motion to take place in the plane XY. Designate forces parallel to the axes, the tangent, and the normal by X, Y, T, N, and let α be the angle (T, X). By projecting we obtain

$$X = T\cos\alpha - N\sin\alpha;\quad Y = T\sin\alpha + N\cos\alpha. \tag{3}$$

Multiplying the first by dx, the second by dy, and adding, we find

$$X\,dx + Y\,dy = T(\cos\alpha\,dx + \sin\alpha\,dy) + N(\cos\alpha\,dy - \sin\alpha\,dx). \tag{4}$$

But the factor of T is equal to ds, and that of N to zero. The amounts of work for the projections of N neutralize each other, though they are not zero separately.

121. The forces of constraint exerted during motion along a line or surface given by its equation are usually said to be due to "geometrical conditions." Under these circumstances there is a normal stress between the surfaces in contact, automatically adjusted so as to produce the requisite local acceleration. This has the definite value $\frac{v^2}{\rho}$ at each point of the guide-curve, and is seen to depend upon the radius of curvature and the speed. The normal stress $\frac{m_1 v^2}{\rho}$ actually existing for a small mass m_1 can be calculated from these data; in one of its aspects it is a force acting upon the moving mass. The function of the geometrical constraints is to bring the normal force to this value. When the surfaces in contact

are plane, the normal force is on the whole zero for the mass; the constraints must just neutralize any other forces acting, that have projections upon the normal. The motion on an inclined plane (§ 120) furnishes a particular example. The projection of the weight m_1g upon the normal is $m_1g \cos i$; and if N is the normal stress between the mass m_1 and the plane, we must have

$$m_1g \cos i + N = m_1 \cdot 0; \quad N = - m_1g \cos i. \qquad (1)$$

When there is sliding contact between surfaces they are not ideally smooth; their relative motion in any direction is opposed by a frictional resistance. It is in many cases a sufficient approximation to experimental data if we treat this type of friction as independent of the relative speed of the two surfaces. Within certain limits we may regard such friction as proportional to normal stress alone; the speed and the area of contact do not appear as determining elements. The range within which this is to be accepted practically must be considered in connection with each problem as it arises. If F is friction-force and N normal stress we may write $F = \phi N$. The factor of N is an empirical constant, and is known as the **coefficient of friction.** It is associated with a pair of surfaces — being influenced by their material and condition — and not with one surface.

Resuming the problem of the inclined plane, let friction proportional to normal stress be introduced as an additional force, while motion occurs along a line of slope, the inclination of the plane to the horizontal being i. The modified equation of motion for the tangent is

$$m_1g \sin i - \phi m_1g \cos i = m_1 \frac{dv}{dt}; \quad g \sin i - \phi g \cos i = \frac{dv}{dt} = \frac{v\,dv}{ds}. \qquad (2)$$

The tangential acceleration remains constant, and the equation kinematical; the results of § 116 are still applicable on substi-

tuting for g the value $(g \sin i - \phi g \cos i)$. There is no difficulty in extending the solution to motion along other lines of the plane.

In what immediately precedes, the tacit supposition is that m_1 continues indefinitely to move downward along the plane. The necessary condition is evidently $m_1 g \sin i \gtreqless \phi m_1 g \cos i$. At the adjustment of the two forces to equality, the speed will be uniform, and $\phi = \text{tg}\, i_1$. This special value of the inclination is termed the "limiting angle" of friction; and the condition named affords an experimental method for determining the coefficient. Regulate i until m_1 slides with constant speed; ϕ is the tangent of the observed inclination. It is found that sliding begins from relative rest at an angle that gives a small positive acceleration. This leads to a distinction between the static and the kinetic coefficient of friction; the former is somewhat larger.

Let it be noticed that friction is a *passive* force merely resisting tendency to motion. In certain circumstances, less than the maximum amount $\phi m_1 g \cos i$ will be called into play, if that is sufficient to prevent motion. For $i < i_1$, the tangential force acting on m_1 is zero; or in magnitude $F = m_1 g \sin i$.

Three different bases for the introduction of friction are indicated in §§ 118, 119, and the present section, as representing experimental results with sufficient approach to truth in large groups of cases. But the questions involved in choosing among the three, or combining them, or excluding them all as inadequate, are to be settled in the full light of experimental investigation in particular subjects. The phenomena of friction are varied and complicated; the one heading includes processes differing widely in character.

122. We shall next examine some other forms in which constraint is exhibited, the motion of a given mass being influenced by connection with other bodies. Let such a system be composed of masses m_1 and m_2 (Fig. 29), hanging freely by a

cord that passes over a smooth fixed cylinder whose axis is horizontal, their weights being the forces active. Suppose $m_1 > m_2$, and call accelerations positive in the direction of the motion that would ensue from a position of rest. The conditions necessitate equality in magnitude for the tangential accelerations of both masses. Hence the equation of motion in the line of the cord is

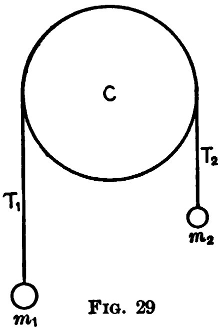


FIG. 29

$$m_1 g - m_2 g = (m_1 + m_2)\frac{dv}{dt}. \qquad (1)$$

Solving for the acceleration, we find

$$\frac{dv}{dt} = \frac{v\,dv}{ds} = \frac{m_1 - m_2}{m_1 + m_2}g. \qquad (2)$$

This acceleration is constant, and the formulas of § 116 are still applicable, if instead of g we write $p = \frac{m_1 - m_2}{m_1 + m_2}g$. The equation of motion is no longer kinematical.

It is instructive to put down the equations of motion for m_1 and m_2 separately, noticing that the constraint due to their connection, which reduces their acceleration to a common value, enters as the stress in the cord. We have, calling the stress $\pm\, T$,

$$m_1 g - T = m_1 p; \quad - m_2 g + T = m_2 p. \qquad (3)$$

The tension is external force as regards m_1 or m_2 singly, but is internal in relation to the system comprising both masses. Accordingly, on adding equations (3) T is removed by cancellation, and equation (1) is reproduced. This illustrates simply and well how the two aspects of a stress may disappear or appear when more or less mass is covered by the equation of motion (§ 60, end). The magnitude of the stress, calculated from either equation, is

$$T = \frac{2\,m_1 m_2}{m_1 + m_2}g. \qquad (4)$$

123. Consider now the resulting motion when the cylinder can revolve freely about its geometrical axis, while the friction at its surface is sufficient to prevent slipping of the cord. If r_1 is the radius of the cylinder, γ angular displacement round its axis, and s vertical displacement of either mass, the condition that the cord shall not slip is expressed by the equation $s = r_1\gamma$. Writing the work equation for the interval $(s = v = 0)$ to (s, v), we find (cf. § 68, (2)), m_3 being the mass of the cylinder (§ 148),

$$(m_1 g - m_2 g)\, s = (m_1 + m_2)\frac{v^2}{2} + \left(\frac{m_3}{2} r_1^2\right)\frac{\omega^2}{2}. \tag{1}$$

But $v = \dfrac{ds}{dt} = r_1\omega$, for no slip; and equation (1) can be differentiated with respect to s, after making that substitution. The result is

$$m_1 g - m_2 g = \left(m_1 + m_2 + \frac{m_3}{2}\right)\frac{v\,dv}{ds} = \left(m_1 + m_2 + \frac{m_3}{2}\right)p. \tag{2}$$

Comparison with equation (1), § 122, shows that in consequence of allowing the cylinder to rotate, half its mass is effectively added to the inertia of the system.

A force-moment is necessary to maintain the angular velocity ω at its value required by the condition $v = r_1\omega$ (§ 73, (1)). Hence we must have $T_1 r_1 > T_2 r_1$ (Fig. 29), or $T_1 > T_2$. It is easy to infer from equation (2) that $T_1 - T_2 = \dfrac{m_3}{2}p$; and this may be proved directly by writing the moment-equation for the rotation, introducing the particular data. We obtain

$$T_1 r_1 - T_2 r_1 = \frac{m_3 r_1^2}{2}\frac{d\omega}{dt}; \quad T_1 - T_2 = \frac{m_3}{2} r_1 \frac{d\omega}{dt} = \frac{m_3}{2}p. \tag{3}$$

The equations of motion for the other parts of the system follow the model of (3), § 122, with T_1 and T_2 replacing T. There are then three simultaneous equations, on adding which the tensions disappear and equation (2) reappears.

If it is desired to take account of a resistance proportional to the speed, the required term can be attached to the equation of motion (2), giving

$$(m_1 - m_2)g - k_1v = \left(m_1 + m_2 + \frac{m_3}{2}\right)\frac{dv}{dt}. \tag{4}$$

Since (4) can be put into the form $A - Bv = \frac{dv}{dt}$, where A and B are positive constants, the process of solution in § 118 can be applied. A correction of weight for buoyancy does not affect the type of equation (4); and this holds true if the assumptions of § 119 are introduced as being nearer the experimental truth. It is plain that the process of the latter section can be extended to this case. With these elements included, the scheme of Atwood's apparatus for determining the weight constant g is fairly well represented in the equation of motion.

124. Let O and Q (Fig. 30) be fixed bearings in which a vertical axis revolves at a constant rate ω. AB is a smooth horizontal tube along the axis of which a small mass m_2 can slide. Another small mass m_1 is threaded without friction on the axis OQ. The two masses are connected by a cord following the line BAQ, and passing over a smooth pin at A; the length of the cord is l. It is required to discuss the equations of motion for the system (m_1, m_2). Let r be the distance of m_2 from A, and for algebraic convenience measure the coördinate s of m_1 positively upward from a point l below A. Then $r = s$ always, and both will be positive. At any position, the

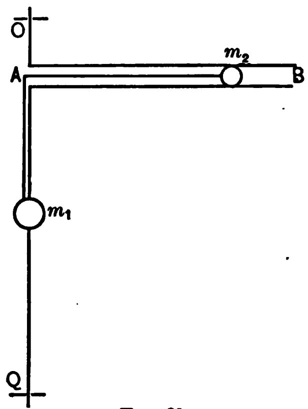


FIG. 30

acceleration of m_1 is $\frac{d^2s}{dt^2}=\frac{d^2r}{dt^2}$; that of m_2 in the line of the cord is $\frac{d^2r}{dt^2}-r\omega^2$; these are not equal, though the cord is of constant length. Introducing the stress in the cord, and writing equations of motion in that line for the masses singly, we have

$$-m_1g+T=m_1\frac{d^2s}{dt^2};\quad -T=m_2\left(\frac{d^2r}{dt^2}-r\omega^2\right). \tag{1}$$

On adding these expressions, the tension is eliminated, and the equation of motion in the line of the cord for the system is

$$-m_1g=m_1\frac{d^2s}{dt^2}+m_2\left(\frac{d^2r}{dt^2}-r\omega^2\right)=(m_1+m_2)\frac{d^2r}{dt^2}-m_2r\omega^2. \tag{2}$$

Equation (2) shows a force (the weight of m_1) distributed between two masses, producing different accelerations in them, and measured by adding together the terms which are products of each mass by its own acceleration.

The resultant force acting upon m_1 is expressed in (1); but another component must be added for m_2. This is in the horizontal plane of its motion, and perpendicular to AB. Its value is (§ 30, (5))

$$P_\gamma=2\,\omega v_r m_2\qquad\left[\omega\text{ constant, }\frac{d\omega}{dt}=0\right]. \tag{3}$$

Equations (2) and (3) together determine the motion of the system. The vertical acceleration of m_2 is zero; its weight is neutralized by the reaction of the tube AB.

We shall consider next the change of kinetic energy in connection with the work equations. The resultant velocity of m_1 is $\frac{ds}{dt}=\frac{dr}{dt}=v_r$; hence for the interval $(r=r_0,\ v_r=v_0)$ to $(r,\ v_r)$ the change in kinetic energy for this mass is $\frac{m_1}{2}(v_r^2-v_0^2)$.

The resultant velocity of m_2 is $\sqrt{v_r^2 + r^2\omega^2}$; and for the change in its kinetic energy we find the expression,

$$\frac{m_2}{2}[(v_r^2 + r^2\omega^2) - (v_0^2 + r_0^2\omega^2)].$$

To obtain the total work for both parts of the system, equations (1) and (3) must be integrated, each with respect to the proper element of distance, and then added. But the stress $\pm T$ contributes nothing to work on the whole; consequently the integral of equation (2) may be substituted for the two integrals of (1). The element of distance for m_1g is $ds = dr$; and we find for the integral between the limits assigned,

$$-m_1g(r - r_0) = (m_1 + m_2)\frac{v_r^2 - v_0^2}{2} - m_2\omega^2\frac{r^2 - r_0^2}{2}. \qquad (4)$$

The element of displacement in the line of P_γ acting upon m_2 at any distance r from A is $rd\gamma = r\omega\, dt$; consequently the integral of (3) with respect to distance is

$$\int_{t_0}^{t} P_\gamma r\omega\, dt = \int_{r_0}^{r} 2\,\omega^2 m_2 r\, dr = m_2\omega^2(r^2 - r_0^2). \qquad (5)$$

Adding this value for the work of P_γ to the work of the weight in equation (4), we obtain, as total work for the system,

$$-m_1g(r-r_0) + m_2\omega^2(r^2-r_0^2) = (m_1+m_2)\frac{v_r^2-v_0^2}{2} + m_2\omega^2\frac{r^2-r_0^2}{2}. \quad (6)$$

The second member of (6) must be the total change in kinetic energy for the two masses; and comparison with the changes in that quantity previously expressed for each mass separately shows the necessary identity.

125. The motion of m_2 is subject to a double constraint: (1) its velocity perpendicular to AB must be accommodated to the constant angular velocity ω; (2) the radial velocity must be common to m_1 and m_2. Each of the constraints is so exer-

cised that work (positive or negative) can be done upon m_2. Under the special conditions $m_1 g = m_2 r\omega^2$, and $v_r = 0$, equation (2), § 124, shows that $\frac{d^2r}{dt^2} = 0$. Hence v_r is zero permanently, and m_2 describes a horizontal circle with centre at A; the inward pull of the cord furnishes exactly the normal force required. This adjustment is unstable; a small displacement inward or outward along AB disturbs balance, and favors its own increase. So long as $v_r = 0$, $P_\gamma = 0$, and the first constraint is not active; but it becomes very prominent if $m_1 g$ is diminished, making the radial force nearly zero. The motion under such circumstances deserves particular attention, because the analysis of it supplies physical explanation for the action of many rotary machines, such as centrifugal pumps and driers. When the radial constraint is of sufficient magnitude to make v_r negative, a contrivance similar to that of § 124 operates as a motor, so long as ω is kept constant.

If we consider m_1 to be removed, and the radial force to be made equal to zero, equation (6), § 124, contains the work of P_γ only, and gives

$$m_2\omega^2(r^2 - r_0^2) = m_2\frac{v_r^2 - v_0^2}{2} + m_2\omega^2\frac{r^2 - r_0^2}{2}; \quad v_r^2 = v_0^2 + \omega^2(r^2 - r_0^2). \quad (1)$$

The latter form is kinematical, and independent of m_2. At the same time the particular suppositions applied to equation (2) of the same section also lead to a kinematical result; for we have

$$0 = m_2\left(\frac{d^2r}{dt^2} - r\omega^2\right); \quad \frac{dv_r}{dt} = r\omega^2. \quad (2)$$

The time-rate of v_r is seen to be positive. We shall suppose $v_r \gtreqless 0$ initially; then it increases at a continually faster *rate* as r grows by the outward motion. The resultant force acting upon m_2 is now P_γ perpendicular to AB; its projections upon

the tangent and normal control the change of speed and the curvature for the path of m_2. Equation (1) shows

$$\frac{m_2}{2}(v_r^2 - v_0^2) = \frac{m_2}{2}(r^2\omega^2 - r_0^2\omega^2). \qquad (3)$$

Hence the work of P_γ is always equally divided between two changes in kinetic energy, one connected with v_r, and the other with the velocity $(r\omega)$ perpendicular to r.

126. In order to lead up to a general conclusion that is of value, let us ascertain the energy relations for a mass m, when AB (§ 124) lies in a plane perpendicular to the vertical axis OQ, but does not intersect it. The line OA drawn perpendicular to AB represents the distance a of the guide from the axis shown at O. The angular velocity is supposed to be constant. The guide AB (Fig. 31) is smooth, and therefore exerts no force in the direction of its own length upon m. Taking a pole at A, with reference line OD, and measuring a coördinate x from A to m, the accelerations parallel and perpendicular to AB of m are seen to be

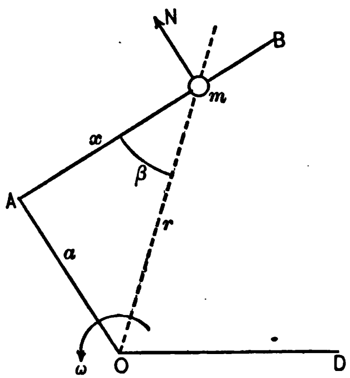


FIG. 31

$$\frac{dv_x}{dt} - x\omega^2 = 0; \quad 2\,\omega v_x \quad \left[\omega \text{ constant, } \frac{d\omega}{dt} = 0\right]. \qquad (1)$$

For origin at O, the accelerations in the same directions are

$$p_1 = \frac{dv_x}{dt} - x\omega^2 = 0; \quad p_2 = 2\,\omega v_x - a\omega^2. \qquad (2)$$

Consequently we find as the equations of motion for m in those lines, N being the force perpendicular to AB,

$$0 = m\left(\frac{dv_x}{dt} - x\omega^2\right); \quad N = m\,(2\,\omega v_x - a\omega^2). \tag{3}$$

The element of displacement in the line of N is $x\omega dt$; hence follows the expression for work within the interval (v_0, x_0) to (v_x, x),

$$\int_{t_0}^{t} Nx\omega\, dt = \int_{x_0}^{x} m\,(2\,\omega v_x - a\omega^2)\, x\omega\, dt. \tag{4}$$

But $v_x dt = dx$; and $x\omega^2 dt = dv_x$, since $\frac{dv_x}{dt} = x\omega^2$ by (3). Making these substitutions, the integration is readily effected, giving

$$\int_{t_0}^{t} Nx\omega\, dt = m\,[\omega^2(x^2 - x_0^2) - a\omega\,(v_x - v_0)]. \tag{5}$$

The resultant force being N, the total change of kinetic energy must be represented in the second member of (5). This may be proved directly by expressing the terminal values E and E_0 of the kinetic energy, and taking their difference. Again referring to the elements A and OD, m has the rectangular component velocities v_x and $x\omega$ respectively along AB and perpendicular to it. Change of origin to O does not affect the latter, and gives for the velocity along AB the value $(v_x - a\omega)$. Using these results, we obtain

$$E - E_0 = \frac{m}{2}\,[(v_x - a\omega)^2 + x^2\omega^2 - (v_0 - a\omega)^2 - x_0^2\omega^2]. \tag{6}$$

But from the first equation of (3),

$$\frac{v_x dv_x}{dx} = x\omega^2; \quad v_x^2 - v_0^2 = \omega^2(x^2 - x_0^2).$$

Introducing these values, expanding, and reducing, we find in corroboration of equation (5),

$$E - E_0 = m\,[\omega^2(x^2 - x_0^2) - a\omega\,(v_x - v_0)]. \tag{7}$$

It should be particularly noted that ω is the angular velocity of the guide AB, and not that of r, the radius-vector drawn from O to m.

In order that m should retain its position relative to AB, while the latter revolves, the mass must describe a circle of radius r with constant angular velocity, and its resultant acceleration must be $r\omega^2$ inward along r. Let β be the angle (x, r); then the guide can compel the component acceleration $r\omega^2 \sin\beta$; but the component $r\omega^2 \cos\beta$ cannot be furnished by a smooth surface. The relative motion of m and AB is determined by the absence of the latter element from the constraint. This remains true when AB is replaced by any smooth curve, other conditions being unaltered. If s_1 is a coördinate measured from a definite point in the curved guide, and $v_1 = \dfrac{ds_1}{dt}$, we have, in parallel with equations (1) of this section and (2), § 125, $\dfrac{v_1 dv_1}{ds_1} = r\omega^2 \cos\beta$. From this follow the relations, first clearing of fractions, and then integrating for the interval (s_0, v_0) to (s_1, v_1),

$$v_1 dv_1 = r\omega^2 \cos\beta\, ds_1 = \omega^2 r\, dr; \quad v_1^2 - v_0^2 = \omega^2 (r^2 - r_0^2). \qquad (8)$$

The remarkable feature of this result is its independence of the particular curve into which the guide is formed. Let any system of smooth guides in the same horizontal plane begin and end on circles of radius r_0 and r, with centres on the axis round which the guides rotate with constant angular velocity. Further, let any (small) masses cross the circle of radius r_0 with equal speeds *relatively to the guides;* then the speeds relative to them will also be equal, when the masses cross the circle of radius r. Compare with this the result of § 120, equation (2); but remark that the latter is true of resultant speeds and fixed curves. The present proposition is established for speed relative to curves that are in uniform rotation

about the same axis normal to their (horizontal) plane (cf. § 111, equation (4)).

127. The work equation, as we have seen in applying it, answers certain types of question by dealing with force through its integrated effects, instead of delaying to formulate the equations of motion. A similar service is rendered by the third fundamental equation, which connects impulse of force with change in momentum. As an example that favors this method of procedure, we select a case of collision between bodies, choosing the simplest conditions in order to bring the matter within our scope. Let two homogeneous spheres, of mass m_1 and m_2, have velocities of translation v_1 and v_2 parallel to the line joining their centres, and come into contact; this is known technically as "central impact" of the spheres. When they collide, the stress between them passes so rapidly from zero through large values to zero again, that the stages cannot be recorded or traced under ordinary conditions. But the impulses are equal and of opposite sign, for the action is always a stress, and the time-limits are the same for both. As a consequence, the changes in momentum are equal and opposite for the two spheres, and the total momentum is unaltered by the collision. Let v_1, v_2, become v'_1, v'_2 when the action ceases; the stress being $\pm P$, and the limits for time 0 and t_1. Then, if it is allowable to disregard all forces but the stress,

$$\int_0^{t_1}(+P)\,dt = m_1(v'_1 - v_1); \quad \int_0^{t_1}(-P)\,dt = m_2(v'_2 - v_2); \quad (1)$$

$$m_1v_1 + m_2v_2 = m_1v'_1 + m_2v'_2. \quad (2)$$

The assignment of positive direction to the force acting on m_1 is arbitrary; but the velocities are to be taken with signs that correspond to this choice.

Equations (1) and (2) represent without limitation the results

for purely internal stresses; and do not imply that the interval is short, except in the neglect of other contributions to the impulses. As an experimental fact, the momentum-changes producible during any interval by a force like weight bear a small ratio to those of forces evoked by collision.

In utilizing equation (2), we are aided by a rule concerning redistribution of momentum that is founded on Newton's investigation in this field; the analysis now current of the process in central impact is really due to him, although the terms of the statement are of later origin. At the surfaces of contact, the deformation or flattening first increases to a maximum, and then in general diminishes. When the maximum is reached, the motion of the bodies relative to each other becomes zero; they are moving with a common velocity. If we consider the entire interval as divided into two parts by the instant of maximum deformation, the ratio between the changes of momentum (impulses) during the first and the second period is found to be nearly constant for the same bodies (or even for bodies of the same materials), although the velocities at collision be varied. This must not be regarded as a mathematical proposition, but as experimental truth. Like other results of similar character, it stands in need of modification or corrective terms outside a certain range, while it affords a sound basis for the introductory treatment of the subject.

128. Let the indicated point of division be t; then the ratio of the impulse for the interval (t, t_1) to the impulse for $(0, t)$ is called the **coefficient of restitution** and denoted by e. As experimentally determined, e is less than unity, and is obtained for materials in pairs. The velocity c common to m_1 and m_2 at the time t is derived from a particular application of equation (2), § 127, for that relation becomes

$$(m_1 + m_2)\,c = m_1 v_1 + m_2 v_2; \quad c = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}. \tag{1}$$

It is plainly apparent that the common velocity c of m_1 and m_2 must be also the velocity at the instant t of the centre of mass for the two masses. But that point must move with constant velocity under the conditions assumed, because internal stresses cannot accelerate it (§ 60). This result can be shown directly in the present instance. If x_1 and x_2 are coördinates of the centres of mass (centres) of the spheres, measured in the line of their motion from any fixed point, the centre of mass for the two bodies has always the coördinate $\bar{x}$ determined by the equation

$$(m_1 + m_2)\bar{x} = m_1x_1 + m_2x_2. \tag{2}$$

Differentiating with respect to time, we find

$$\frac{d\bar{x}}{dt} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2} = c. \tag{3}$$

During the second part of the entire interval the changes in momentum are

$$m_1(v'_1 - c) = em_1(c - v_1); \quad m_2(v'_2 - c) = em_2(c - v_2). \tag{4}$$

Cancellation of mass-factors and elimination of c by subtraction shows $v'_1 - v'_2 = -e(v_1 - v_2)$. This aspect of the action during collision is contained in the statement: By central impact the velocity of one body relative to the other is reversed in sign; and its magnitude is reduced in the ratio of e to unity.

In order to ascertain the relation between the final velocities and v_1, v_2, substitute in (4) the value of c from (1). We obtain

$$\left.\begin{aligned} v'_1 &= \frac{m_1v_1 + m_2v_2}{m_1 + m_2} - e\,\frac{m_2(v_1 - v_2)}{m_1 + m_2}; \\ v'_2 &= \frac{m_1v_1 + m_2v_2}{m_1 + m_2} + e\,\frac{m_1(v_1 - v_2)}{m_1 + m_2}. \end{aligned}\right\} \tag{5}$$

All of these results may be extended to the rebound of a sphere

m_1 from the surface of any very large mass m_2, by giving the value zero to the ratio $\frac{m_1}{m_2}$.

Some cases that present themselves in practice indicate vanishing values of the coefficient e. As we approach that limit, the interval (t, t_1) drops out of consideration — the impact is termed "inelastic"; the masses remain in contact, and move with the velocity c.

It is an accompaniment of collision that the original kinetic energy of the masses is in part transformed, giving rise to phenomena of heat, light, sound, and other forms of vibration, as well as mechanical effects like crushing and splitting. In this sense kinetic energy is *lost* when a blow is struck. The amount of energy L that disappears thus is readily calculated from the expressions already obtained; we find

$$L = \left(\frac{m_1 v_1^2 + m_2 v_2^2}{2}\right) - \left(\frac{m_1 v'^2_1 + m_2 v'^2_2}{2}\right)$$

$$= (1 - e^2)\frac{m_1 m_2}{2(m_1 + m_2)}(v_1 - v_2)^2. \qquad (6)$$

129. When conditions are permanent (*i.e.* for steady motion) the impulse equation can be written (eq. (5), § 54),

$$P(t - t_0) = m(v - v_0); \quad P = \frac{m(v - v_0)}{t - t_0}. \qquad (1)$$

Where the data enable us to give expression to the change per second of momentum in a given line, that rate measures the force active in the line. By way of applying this idea, let a stream of water in the form of a cylinder with its axis vertical flow steadily down against a horizontal plate, and run off radially with horizontal velocity. If the cross-section of the stream has an area a, and v is the velocity with which it reaches the plate, the change per second in vertical momentum is $q(0 - v) = -qv$, denoting by q the mass per second of water

that arrives at the plate. Hence the constant force P exerted *upon the plate* by the continuous process of impact is qv. To express this in terms of the data, we have, δ being the density of the water, $q = \delta av$; $P = \delta av^2$. We observe represented in this result the fact that such a force varies as the square of the speed with which the jet strikes. All gas-pressure has its origin in such momentum-changes, if the kinetic view of the matter is truth.

In further illustration of the same method, consider water running in a nearly horizontal trough $ABCD$ (Fig. 32), which is so curved in the part BC that the tangents CD and AB make an angle α. Let the speed v be constant everywhere, the cross-section being negligible in comparison with the radius of curvature between B and C. It is required to calculate the total force parallel to AB exerted by the water upon the curve BC. With notation as before, the change per second of momentum parallel to AB in the water is $q(v \cos\alpha - v) = qv(\cos\alpha - 1) = \delta av^2(\cos\alpha - 1)$; and the force active upon the curved part of the trough is found by reversing the sign. This value may be verified by expressing the normal force at each point, and projecting parallel to AB; the tangential force is zero for constant speed. We have, using α temporarily as variable,

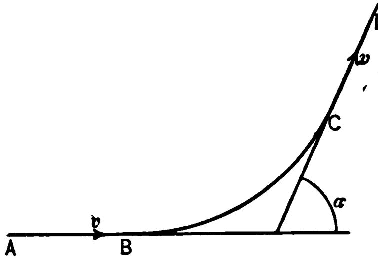


FIG. 32

$$dN = -dm\frac{v^2}{\rho}; \quad dP = -dm\frac{v^2}{\rho}\sin\alpha; \quad dm = \delta a\rho d\alpha; \tag{2}$$

$$P = -\int^{\alpha} \delta av^2 \sin\alpha\, d\alpha = \delta av^2(\cos\alpha - 1). \tag{3}$$

130. In various contrivances that are typically represented by "reaction-wheels" and rockets, internal stresses are made use of to produce motion of certain portions relative to the remainder, those parts being thrown off, while force is exerted upon the rest of the original system. It is an essential condition governing all such processes, that the total momentum remains constant so far as the effects of such stresses are concerned (eq. (2), § 127); though that quantity may be changed by external forces acting simultaneously or afterward. From the rate of change in momentum, it is often practicable to determine the force at the surface where separation occurs, especially when such motions have become steady. The rate in question is connected with the velocity of the two portions relative to each other, because each element of mass thrown off was originally moving with the same velocity as the rest.

Conceive a cylindrical rocket, of mass m_0 at $t = 0$, to be arranged for projection vertically upward in a vacuum, with no external force acting but weight. Let q be the mass per second of gas issuing centrally from the lower base, with velocity $-v_1$ relative to the orifice, the upward vertical being positive. Then qv_1 is the force instantaneously exerted upon the rocket itself. For, if m is its mass at any time t, and v its (vertical) velocity, the momentum in this part of the system is mv. Again, the momentum of the gas that has been ejected is $\int_0^t q(v - v_1)\,dt$, apart from influences exerted by other bodies. Consequently, under the action of internal stresses alone, we should have, if the whole system were originally at rest,

$$mv + \int_0^t q(v - v_1)\,dt = 0. \qquad (1)$$

This is a general relation, and may be differentiated with respect to time, giving

$$\frac{dm}{dt}v + m\frac{dv}{dt} + q(v - v_1) = 0. \qquad (2)$$

But $$m = m_0 - \int_0^t q\,dt; \quad \frac{dm}{dt} = -q; \tag{3}$$

and consequently the result follows:

$$m\frac{dv}{dt} - qv_1 = 0. \tag{4}$$

If now in addition weight is acting, and attention is directed to the rocket, not to the gas that has issued, the external force adds vertical momentum at the rate $-mg$, and the equation of motion for m becomes

$$qv_1 - mg = m\frac{dv}{dt}. \tag{5}$$

So long as $qv_1 < mg$, the rocket cannot ascend; it presses downward upon its support.

131. In forming the integral for the work of a stress $\pm P$ that acts between two masses, the force P is multiplied by an element of distance that is the difference of the displacements in the line of stress at its ends. The work done upon m_1 may be written $(+P)\,ds_1$; that done upon m_2 is then $(-P)\,ds_2$; and the total work-element is $P(ds_1 - ds_2)$. The work is the product of the force-magnitude and the amount by which the line lengthens or shortens; the signs follow the usual rule.

If the bodies move in a straight line with translation, and from rest, under the influence of such a stress, the change in kinetic energy measuring the work is divided between them in the inverse ratio of the masses. For we have

$$\frac{m_1v_1^2}{m_2v_2^2} = \left(\frac{m_1v_1}{m_2v_2}\right)^2 \cdot \frac{m_2}{m_1} = \frac{m_2}{m_1}. \tag{1}$$

This is true, not only for the entire interval, but for any part of it. Let v'_1, v'_2 be any other pair of values; then equally for

them $\frac{m_1 v'^2_1}{m_2 v'^2_2} = \frac{m_2}{m_1}$; and because the quantities are in equal ratio,

$$\frac{m_1(v_1^2 - v'^2_1)}{m_2(v_2^2 - v'^2_2)} = \frac{m_2}{m_1}. \tag{2}$$

Consequently the entire kinetic energy appears in m_1 in proportion as $\frac{m_1}{m_2}$ approaches zero (cf. § 112). The conclusion of § 130 shows how the kinetic energy may be also completely diverted from one mass, in that type of action, by an external force.

132. When one body concerned in an impact is constrained to turn about a fixed axis, the total momentum cannot be assumed constant, because the condition of freedom from external force applied at the axis is not in general fulfilled. An example is added of the reasoning that guides the solution of such problems.

Let a homogeneous rectangular prism ABD (Fig. 33) turn freely about a horizontal axis parallel to one edge, and lying in a plane of symmetry. The prism is at rest with its centre of mass C vertically below the axis shown at O. A small mass m moves in the (vertical) plane of the diagram, and strikes the face AB when the line of its velocity v is horizontal, and distant h from O. The effect of the impact upon the motion of the two bodies is to be examined, under the supposition that no noticeable displacement is produced during the time covered by the impulse.

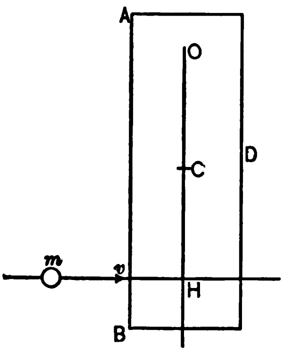


FIG. 33

Call the stress $\pm P$, the upper sign being taken for the prism, whose moment of inertia for the axis at O we denote by I_O.

Regarding h as constant, the moment-equation and its time-integral (eq. (1), (2), § 73) become, for $\omega = 0$, at $t = 0$,

$$hP = I_O \frac{d\omega}{dt}; \quad h\int_0^{t_1} Pdt = I_O\omega. \tag{1}$$

Writing the impulse equation for m, and multiplying it by h, we obtain

$$-h\int_0^{t_1} Pdt = hm(v' - v). \tag{2}$$

Adding (1) and (2), the resultant equation is

$$hmv = I_O\omega + hmv'. \tag{3}$$

This relation may be announced in the form: The moment of momentum for the axis of rotation remains constant. It presents the principle that is dominant in problems of this type. The conclusion would not be affected, as regards this statement of it, if the initial angular velocity of the prism were different from zero, since we should then find

$$hmv + I_O\omega_0 = I_O\omega + hmv'. \tag{4}$$

At the position assumed for the impact, the moment of weight is zero. But when the angular velocity imparted carries OC away from the vertical, the weight of the prism does work which reduces the kinetic energy $I_O \frac{\omega^2}{2}$. Let m_1g be that weight, and γ the angle made with the vertical by $OC = \bar{r}$. Unless ω is of such magnitude that the prism makes complete revolutions, the amplitude of the swing is determined by the relation,

$$m_1g\bar{r}(1 - \cos\gamma_1) = I_O \frac{\omega^2}{2}. \tag{5}$$

Observation of γ_1 enables us to calculate ω, and the effect of a moment of impulse in producing rotation.

As a particular value, we can make $h = \frac{I_O}{m_1\bar{r}}$, giving $\frac{I_O}{h} = m_1\bar{r}$.

On dividing equation (4) by h, and substituting according to the above relation, the moment of inertia is eliminated and the simplified equation remains,

$$mv + m_1\bar{r}\omega_0 = mv' + m_1\bar{r}\omega. \tag{6}$$

But $m_1\bar{r}\omega_0$ and $m_1\bar{r}\omega$ are total momentum for the prism, at $t = 0$, and $t = t_1$, since the factors of m_1 are velocities of C (§ 61). Hence for this special selection of h, we have constant momentum in the line of v coexisting with constant moment of momentum. We must interpret this to mean that no external force parallel to v is necessary, when h has the value indicated, to constrain the prism to begin rotating about the axis at O. That is, the force P itself then produces zero acceleration at O in the line of v. Simple analysis of the conditions shows this to be the fact. The moment of P to set up rotation about an axis at C parallel to the one at O is, for this choice of h,

$$\left(\frac{I_O}{m_1\bar{r}} - \bar{r}\right)P = \left(\frac{I_O - m_1\bar{r}^2}{m_1\bar{r}}\right)P = \frac{I_C}{m_1\bar{r}}P. \tag{7}$$

The moment-equation for the rotation relative to the centre of mass (§ 80) becomes

$$\frac{I_C}{m_1\bar{r}}P = I_C\frac{d\omega}{dt}; \quad \frac{d\omega}{dt} = \frac{P}{m_1\bar{r}}. \tag{8}$$

The corresponding linear acceleration at O being $\bar{r}\dfrac{d\omega}{dt}$, we find for its value $\dfrac{P}{m_1}$; it is parallel to v. The acceleration for the translation with the centre of mass is seen to be in the same line at O, but in the opposite direction; and its magnitude is also $\dfrac{P}{m_1}$. Hence the two elements of acceleration neutralize each other at O.

The actual rotation occurs about a principal axis for the point O, as considerations of symmetry show (§ 151); therefore no directive couples are needed (§ 82). The force P lies

in a plane perpendicular to the rotation-axis; hence no constraint parallel to that axis can be required. In the plane of the diagram, we have just proved no constraint parallel to P to be active; and there are left only the forces $m_1\bar{r}\omega^2$ and weight in the line CO. When the force P is so adjusted in position and direction as to cause rotation with such minimized constraint, the point (H in Fig. 33) at which its line of action intersects the plane containing the axis and C is termed the **centre of percussion**. The centres of oscillation and percussion coincide for the data assumed here (§ 97).

133. Let a rectangular prism $ABFH$ (Fig. 34), having a vertical plane of symmetry shown in the diagram, be acted upon by its weight, and guided by smooth horizontal and vertical planes XZ and YZ, so that its centre of mass C describes a curve in the plane XY. The origin is at O, the coördinates of A and B are x and y, those of C, $\bar{x}$, $\bar{y}$; the edges AH, AB, are $2b$, $2a$; γ is the angle made by AH with X. The motion at any value of γ between 0 and $\frac{\pi}{2}$ is to be described, using the conception of instantaneous centre (§ 21). That point can be located by drawing perpendiculars to the velocities of two points in the body; B and A are convenient here; the centre I is thus determined. We see directly that the locus of I (the fixed centrode) is a circle with centre at O, and radius $2a$. Since BIA remains a right-angled triangle, with constant hypotenuse AB, the moving centrode is a circle of diameter $2a$, which rolls inside the

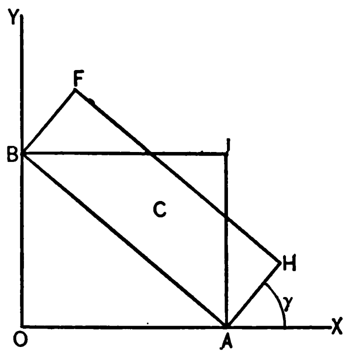


FIG. 34

larger one, I being always the contact-point, and having coördinates x, y.

With the ordinary notation for initial values, and writing m_1 for the mass of the prism, the work equation takes the form,

$$-m_1 g(\bar{y}-\bar{y}_0)=I_I\frac{\omega^2-\omega_0^2}{2}. \tag{1}$$

Let the variable distance IC be $\bar{r}$; then at any position $I_I=I_C+m_1\bar{r}^2$ (§ 150). The value of I_C is $m_1\frac{a^2+b^2}{3}$. From the geometrical relations we find:

$$\left.\begin{aligned}x^2+y^2&=4a^2\\ \operatorname{tg}\gamma&=\frac{x}{y}\end{aligned}\right\};\quad \left.\begin{aligned}\bar{y}&=a\cos\gamma+b\sin\gamma\\ \bar{x}&=a\sin\gamma+b\cos\gamma\end{aligned}\right\};\quad \bar{r}^2=(x-\bar{x})^2+(y-y)^2. \tag{2}$$

Equations (2) in connection with (1) make ω a known function of γ, so that all velocities become assignable in terms of position with respect to I. The total kinetic energy is given by the second member of (1). This can be separated into energy $I_C\frac{\omega^2}{2}$ due to rotation about an axis normal to XY at C, and energy $m_1\frac{\bar{r}^2\omega^2}{2}$ connected with translation, the velocity being $\bar{r}\omega$ perpendicular to IC.

134. The diagram (Fig. 35) shows a vertical section through the centre of mass C of a cylinder whose axis is horizontal, and a fixed plane AB of inclination β. The equations of motion for the standard elements of translation and rotation are to be derived, when the

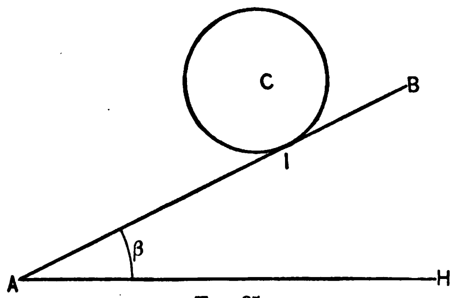


Fig. 35

cylinder is constrained to roll (without slipping), the path of C being parallel to a line of slope.

The instantaneous axis, shown at I, is horizontal. The kinetic energy for the rotation about it can be calculated for any position from the work equation for the interval (s_0, ω_0) to (s, ω),

$$-m_1 g(s - s_0)\sin\beta = I_I \frac{\omega^2 - \omega_0^2}{2}. \tag{1}$$

Here m_1 is the mass of the cylinder, and s a coördinate measured positively from A toward B. I_I is a constant magnitude, because $I_I = I_C + m_1\bar{r}^2$, and $\bar{r}$ is now the radius of the cylinder which we shall write r_1. The value of $I_C = \frac{m_1 r_1^2}{2}$, and $r_1^2\omega^2 = v^2 = \left(\frac{ds}{dt}\right)^2$ at C. Make these substitutions, and differentiate with respect to s. The acceleration of C is determined by the resulting equation,

$$-m_1 g \sin\beta = \tfrac{3}{2} m_1 \frac{v\,dv}{ds}; \quad \frac{dv}{dt} = -\tfrac{2}{3} g \sin\beta. \tag{2}$$

The acceleration of C due to weight alone would be $-g\sin\beta$. Hence additional force parallel to AB must be active; $\frac{1}{3} m_1 g \sin\beta$ in magnitude, and positive. This is the constraint that acts upon the cylinder at its contact with the plane, and causes rolling; it may be exercised as frictional resistance to slipping, or in other ways.

The equation of motion for rotation about the cylinder-axis is (§§ 73, 77)

$$r_1\left[\frac{m_1}{3} g \sin\beta\right] = \frac{m_1 r_1^2}{2}\frac{d\omega}{dt}; \quad \frac{d\omega}{dt} = \tfrac{2}{3}\frac{g}{r_1}\sin\beta. \tag{3}$$

This is in accord with the value of $\frac{dv}{dt}$ in (2); since, by the convention established for positive angular velocity, $v = \frac{ds}{dt} = -r_1\omega$.

When friction is relied upon to establish the constraint, and

make the cylinder roll, the limiting angle (§ 121) at which it ceases to be effective is $\beta_1 = \text{arc tg}\,(3\,\phi)$, as appears by solving the equation,

$$\phi\, m_1 g \cos \beta_1 = \frac{m_1}{3} g \sin \beta_1. \qquad (4)$$

135. A plane lamina of mass m rotates at the rate ω about an axis perpendicular to its plane. When its thickness can be disregarded, the resultant of the *normal forces* for all its parts is a force acting through its centre of mass C toward the axis, and having the magnitude $m\bar{r}\omega^2$; $\bar{r}$ meaning as usual the distance from the axis to C. In order to prove this auxiliary proposition, take axes X and Y in the plane of the lamina, and intersecting upon the rotation-axis at O. Let x, y, r, be coördinates and radius-vector for any differential mass dm. The element of normal force is $-\,dmr\omega^2$; projecting it upon X and Y, we obtain $-\,dmx\omega^2$, $-\,dmy\omega^2$; and integrating to cover the entire mass, we have

$$N_x = -\,\omega^2 \int_0^m x dm = -\,\omega^2 \bar{x} m; \quad N_y = -\,\omega^2 \int_0^m y dm = -\,\omega^2 \bar{y} m. \quad (1)$$

The resultant of these passes through O, because all the elements do so. The magnitude of the resultant is $-\,m\bar{r}\omega^2$, with minus sign because both projections are negative. Its inclination to the X axis is given by $\text{tg}\,\alpha = \frac{\bar{y}}{\bar{x}}$; therefore it acts in the line CO. This resultant of normal forces that depend upon angular velocity can be considered apart from any question of tangential forces and angular acceleration. Such resultants are important elements in problems involving rotation; and the proposition just proved affords a clew to their magnitude and position (*i.e.* force and force-moment, §§ 79, 80) in many cases where regular solids revolve about fixed axes. We shall illustrate this with some instances.

Let a spherical mass m_1 be in rotation with angular velocity

ω about any axis from which the distance of its centre is $\bar{r}$. The centres of its circular sections by planes perpendicular to the rotation-axis lie on a diameter parallel to that line. The normal forces for each section yield a resultant in the plane of these parallels, at right angles to the axis, and $-dm\bar{r}\omega^2$ in magnitude. The force of the resultant when the entire sphere is included will be $-\int_0^{m_1} dm\bar{r}\omega^2 = -m_1\bar{r}\omega^2$; since $\bar{r}\omega^2$ is a common factor. Its direction is parallel to that of the elements that compose it. The moment of the resultant is zero for any axis passing through the centre of the sphere, on account of the symmetrical arrangement of equal sections. For our present purpose, the sphere can be treated as a particle of mass m_1 at its centre.

A similar conclusion can be drawn wherever the body has a plane of symmetry perpendicular to the axis. The sections of the body parallel to that plane can always be grouped in pairs that are equidistant from it, and equal in every respect. The addition two by two of their resultants for normal forces gives a force in the plane of symmetry, and zero moment about any axis lying in that plane. But such pairs exhaust the entire mass; and the proposition proved for the lamina covers the final combination of the partial resultants in the plane of symmetry. Consequently the resultant of normal forces in such a case is always a force $m_1\bar{r}\omega^2$ acting through the centre of mass toward the rotation-axis. By comparing this conclusion with equations (7) and (8), § 82, we see that the directive couples of the resultant, needed in general when Z is a (fixed) axis of rotation, vanish if the supposed symmetry exists. And Z is a principal axis for the origin, when the integrals involved in those equations are zero (§ 151).

136. Returning to the case of a sphere, we shall examine more particularly two types of mechanical contrivance that

secure pure rotation. Let a fixed vertical axis OQ (Fig. 36) be given, to which a sphere with its centre at C is attached by an arm AC whose mass we shall not consider. There is a hinge at A allowing change in the angle CAQ. Let m be the mass of the sphere, a the length CA, γ the angle CAQ, and ω the angular velocity about OQ. Required the position of such adjustment between ω and γ that C shall describe a horizontal circle round the axis. Under those conditions ω must be constant, and there is no tangential force; the complete resultant for the sphere is $-m\bar{r}\omega^2$, DC being $\bar{r}$. On account of the hinge, the force exerted by the arm must be directed along CA. Its vertical projection neutralizes the weight mg, while its horizontal part supplies the necessary constraint at C. Denoting the tension in CA by T, these ideas find expression in the equations,

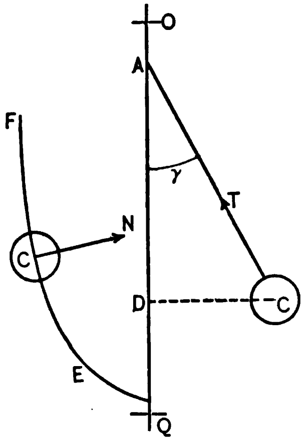


FIG. 36

$$T\cos\gamma + mg = 0; \quad T\sin\gamma = -m\bar{r}\omega^2. \tag{1}$$

After eliminating T, we find as equation of condition for the adjustment,

$$\operatorname{tg}\gamma = \frac{\bar{r}\omega^2}{g}; \quad \cos\gamma = \frac{g}{a\omega^2}. \tag{2}$$

This arrangement is known as a "conical pendulum." The period for one revolution is

$$T = 2\pi\sqrt{\frac{a\cos\gamma}{g}}. \tag{3}$$

A more general form of constraint is also shown in Fig. 36, where EF is any smooth guide in the vertical plane. Equations (1) continue to apply, if instead of T we introduce N, the force exerted at C in the normal to the guide; and provided that we interpret γ as the angle between N and OQ. The general condition of adjustment for the circular path of C remains as before $\operatorname{tg}\gamma = \frac{\bar{r}\omega^2}{g}$. There will be this relation at every position of C on such a guide, for given values of ω and g, when the equation of its curve satisfies the relation $\frac{\operatorname{tg}\gamma}{\bar{r}} = \frac{\omega^2}{g}$. If the axis of a parabola is directed vertically upward and taken as X, we have $y^2 = 2px$; $\frac{dx}{dy} = \frac{y}{p}$. But $\frac{dx}{dy} = \operatorname{tg}\gamma$, $y = \bar{r}$, for the case in question; and the parabola as a guide will fulfil the requirement; but we must make $p = \frac{g}{\omega^2}$. The surface of water in a steadily rotating vessel exemplifies this.

137. If for the sphere C and arm CA (Fig. 36) we substitute a cylinder whose axis is OX_1, we obtain the arrangement represented in a section containing the vertical rotation-axis OX by Fig. 37. The axis of the hinge is the horizontal diameter of the upper base shown at O; and the equation of condition is to be established for given constant value of γ, the angle X_1OX

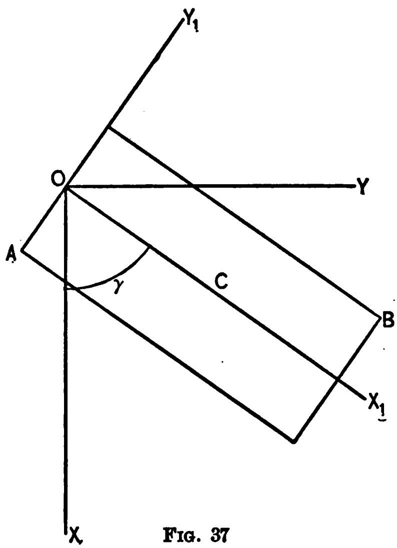


FIG. 37

There is now no plane of symmetry perpendicular to X, and the integrals corresponding in type to those of equations (7) and (8), § 82, cannot be assumed to vanish for the origin O. Noticing that the rotation-axis is here X, and $\frac{d\omega}{dt} = 0$, the integrals that we must consider are seen to be

$$M(y) = \omega^2 \int_0^{m_1} xz\, dm; \quad M(z) = -\omega^2 \int_0^{m_1} xy\, dm. \tag{1}$$

In every azimuth round X the momentary choice of Y and Z is the same *relatively to the cylinder;* hence there is symmetry with regard to every plane XY, $\int_0^{m_1} xz\, dm$ is always zero, and $M(y)$ vanishes. The remaining integral can be transformed by using the coördinate-relations,

$$x = x_1 \cos\gamma - y_1 \sin\gamma; \quad y = y_1 \cos\gamma + x_1 \sin\gamma. \tag{2}$$

Substitution of these values yields

$$xy = (x_1^2 - y_1^2)\sin\gamma\cos\gamma + x_1 y_1 (\cos^2\gamma - \sin^2\gamma); \tag{3}$$

but the mass-integral of the last term in (3) disappears, because X_1Z is a plane of symmetry. Therefore the final result is

$$\begin{aligned} M(z) &= -\omega^2 \int_0^{m_1} xy\, dm = -\omega^2 \sin\gamma\cos\gamma \int_0^{m_1} (x_1^2 - y_1^2)\, dm \\ &= -\omega^2 \sin\gamma\cos\gamma \left[\int_0^{m_1} (x_1^2 + z^2)\, dm - \int_0^{m_1} (y_1^2 + z^2)\, dm\right] \\ &= -\omega^2 \sin\gamma\cos\gamma\,(I_{Y_1} - I_{X_1}). \end{aligned} \tag{4}$$

In equation (4) the moment-term is evaluated for the resultant of a system of normal forces $-dm\, r\omega^2$ throughout the mass m_1 of the cylinder. The complete expression of the resultant must include this moment about Z at every position; and the external forces must be so applied as to exercise it, while rotation about the fixed axis X proceeds at the constant rate ω. If the cylinder were *rigidly* attached to OX in any way, the

bearings of the rotation-axis might furnish the supplementary constraints (§ 82) needed in order to bring the moment exactly to the required amount for all values of γ and ω. The peculiarity of the present contrivance lies in the hinged connection at O, which renders it impossible for the axis to exert constraint in the form of moment about Z. Consequently we must rely upon the other external forces to do this; and the weight-moment produces the requisite effect when the equation is satisfied,

$$-m_1 g\bar{y} = -\omega^2 \sin\gamma\cos\gamma\,(I_{Y_1} - I_{X_1}). \qquad (5)$$

For $\gamma = 0$, $M(z) = 0$ at all values of ω. If $I_{Y_1} = I_{X_1}$, $M(z) = 0$ at all values of γ and ω; adjustment of the weight-moment to the value required occurs at $\gamma = 0$ or π only. The moments of inertia in the case of a cylinder are, in terms of mass (m_1), radius (r_1), and altitude (h),

$$I_{Y_1} = m_1\left(\frac{r_1^2}{4} + \frac{h^2}{3}\right); \quad I_{X_1} = m_1\frac{r_1^2}{2}. \qquad (6)$$

These become equal when $\frac{h^2}{3} = \frac{r_1^2}{4}$; $h = \frac{r_1}{2}\sqrt{3}$. So long as $I_{Y_1} > I_{X_1}$, there is in general a value of γ between 0 and $\pm\frac{\pi}{2}$ that gives adjustment. Such values lie in the interval $(\pm\frac{\pi}{2}, \pi)$, if $I_{Y_1} < I_{X_1}$. Therefore the constraint will always oppose coincidence of the rotation-axis, and the axis for which moment of inertia is greater.

138. Water exerts horizontal pressure due to its weight normally upon one vertical face of a rectangular prism whose upper horizontal edge lies in the water-surface; the resultant of the total water pressure is to be determined. Let the area in contact with the water have width b and depth a. Then, if δ is the density of water, the force acting upon the horizon-

tal strip bdx at any depth x is known to be $\delta gx \cdot bdx$. The total force against the area ab is

$$P = \delta bg \int_0^a x\, dx = \frac{\delta bga^2}{2}. \tag{1}$$

We conclude from the symmetrical distribution of pressure that the axis for the total moment must be parallel to the line in which the water-surface intersects the prism. If that line be chosen as axis, the lever-arm for each element of moment is x; and the total moment,

$$M = \delta bg \int_0^a x^2\, dx = \frac{\delta bga^3}{3}. \tag{2}$$

Expressing the resultant as a force of definite position, it must have the magnitude determined by equation (1), and act normally to the face ab in its vertical median line, at a distance x_1 below the surface such that

$$x_1 = \frac{M}{P} = \frac{\frac{1}{3}\,\delta bga^3}{\frac{1}{2}\,\delta bga^2} = \tfrac{2}{3}\,a. \tag{3}$$

If M had been calculated for any other (parallel) axis, the relation would still be $Px_1 = M$, but x_1 is measured from the axis assumed for M; x_1 is an average lever-arm for the forces, and must locate the same point, no matter how calculated (cf. § 146). Using the other plan of specifying a resultant as a force at the centre of mass and a couple-moment, the force is P as before; and denoting the couple-moment by M_1, we have the relation (§ 80)

$$\frac{a}{2} \cdot P + M_1 = M; \quad M_1 = \tfrac{1}{12}\,\delta bga^3. \tag{4}$$

Here $\frac{a}{2}$ is typical of the distance (measured in a line perpendicular to P) from the centre of mass to the axis for M.

Supposing that M_1 is to be produced by P itself, acting "off centre," the force must be moved down through a distance

$$\frac{\frac{1}{12}\,\delta bga^3}{\frac{1}{2}\,\delta bga^2} = \tfrac{1}{6}\,a; \text{ and } \tfrac{1}{6}\,a + \tfrac{1}{2}\,a = \tfrac{2}{3}\,a \text{ as before.}$$

The equilibrant (§ 86) of the water pressure must be equivalent to a force $-P$ at the centre of mass, and a couple-moment $-M_1$. The latter may be due to $-P$ itself; and in that case $-P$ must act oppositely in the *same line* as $+P$ when the latter is the resultant.

139. Consider the conditions that the edge A (Fig. 34, page 225) of the rectangular prism AHB may be on the point of sliding outward along X under the influence of its weight and friction against the planes, XZ, YZ. With such a symmetrical distribution over the area of contact, forces may be concentrated upon the median plane represented in the diagram, and regarded as acting there. Let N_1, N_2, be the forces exerted upon the block normal to OX, OY, at A and B. Then the frictions along OX and OY can be assumed as ϕN_1 and ϕN_2 respectively. The equilibrium equations for force in this case are

$$-m_1 g + N_1 + \phi N_2 = 0; \quad -\phi N_1 + N_2 = 0. \tag{1}$$

Any rotation to be taken account of would occur about an axis parallel to Z, say at A. The force-moment equation for equilibrium, when sliding is just about to begin, gives the relation

$$m_1 g(a \sin\gamma - b\cos\gamma) - 2\,N_2\,a\cos\gamma - 2\,\phi N_2\,a\sin\gamma = 0. \tag{2}$$

These three equations serve to determine the three unknown quantities N_1, N_2, γ.

140. A flexible rope is carried round a curved surface, the element of friction at each point being proportional to the normal pressure there. The equation is to be found that con-

nects the tensions T_1 and T_0 at the ends of the curve, when the rope can just slide in the direction of T_1.

Draw tangents at the extremities of any arc Δs of the curve, and let their difference of direction be $\Delta\alpha$. The tensions at the points of tangency being T and $T+\Delta T$, and the coefficient of friction ϕ, the inequality subsists, if ΔF is the actual friction on the arc,

$$2\,\phi(T+\Delta T)\sin\frac{\Delta\alpha}{2} > \Delta F > 2\,\phi T\sin\frac{\Delta\alpha}{2}. \qquad (1)$$

Dividing by $\Delta\alpha$, we obtain

$$\phi(T+\Delta T)\frac{\sin\frac{\Delta\alpha}{2}}{\frac{\Delta\alpha}{2}} > \frac{\Delta F}{\Delta\alpha} > \phi\, T\frac{\sin\frac{\Delta\alpha}{2}}{\frac{\Delta\alpha}{2}}. \qquad (2)$$

The limiting values of the extreme terms coincide when $\Delta\alpha = \Delta T = 0$, and an equation results, giving $\frac{dF}{d\alpha} = \phi T$. For incipient sliding we must have at each point, $T = F + T_0$, where F is the total, up to that point, of friction to be overcome. Hence $dT = dF$, and $\frac{dT}{d\alpha} = \phi T$. The equation last written can be integrated for the interval $(0, T_0)$ to (α_1, T_1), and leads to the relations,

$$\log\left(\frac{T_1}{T_0}\right) = \phi\alpha_1; \quad T_1 = T_0 e^{\phi\alpha_1}; \quad F_1 = T_1 - T_0 = T_0(e^{\phi\alpha_1} - 1). \quad (3)$$

These are dependent upon total curvature, but contain no reference to the particular curve.

141. A heavy flexible rope, the dimensions of whose cross-section are negligible in comparison with its length, hangs in equilibrium between two points A and B in the same horizontal line at a distance $2a$. The equation of the curve that it assumes is to be derived, first on the supposition that its weight is distributed uniformly along its horizontal projection.

Take the lowest point O (Fig. 38) as origin, and the hori-

zontal tangent as X; let Q be any point (x, y). The part OQ itself is in equilibrium, the external forces being the tension H_1 at O, the tension T at Q, and the weight of OQ. The force

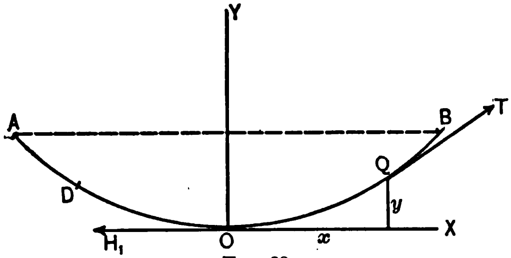


FIG. 38

last named is ϵx, if ϵ is weight per unit length of horizontal coördinate. Write H and V for the horizontal and vertical projections of T; then we can conclude $H_1 + H = 0$; $V + \epsilon x = 0$. But Q is any point; hence the horizontal projection of the tension has the same value everywhere. At Q we have for the inclination of the tangent to OX,

$$\operatorname{tg} \alpha = \frac{V}{H} = \frac{\epsilon x}{H_1} = \frac{dy}{dx}. \tag{1}$$

Integrating with initial conditions $\alpha = 0$, $x = y = 0$, we find

$$y = \frac{\epsilon}{2 H_1} x^2. \tag{2}$$

The curve is identified as a parabola with vertex at O, and axis directed vertically upward. To determine H_1, let it be given that O is b below A or B. Substituting in equation (2), we find $H_1 = \frac{\epsilon a^2}{2b}$. The tensions T_1 at A and B are in magnitude $T_1 = \sqrt{H_1^2 + \epsilon^2 a^2}$; and for those points we have $\operatorname{tg} \alpha_1 = \frac{\epsilon a}{H_1}$.

Secondly, let the distribution of weight be uniform along the arc of the curve. Changing equation (1) to correspond, it

becomes $\dfrac{dy}{dx} = \dfrac{\epsilon s}{H_1}$, if we measure s from O. For convenience, introduce $c = \dfrac{H_1}{\epsilon}$; the equation $\dfrac{dy}{dx} = \dfrac{s}{c}$ is recognized as belonging to the catenary. The origin for this curve is taken at the distance c below O; and the serviceable relations are

$$y^2 = c^2 + s^2; \quad y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right). \tag{3}$$

In the first equation of (3), write for c its value and clear of fractions. From the equation $(\epsilon y)^2 = H_1^2 + (\epsilon s)^2$, we discover that the tension at any point is proportional to the ordinate.

When the rope is hung from points, such as B and D, that are not in a horizontal line, the problem can be treated by regarding the curve as completed to A, and then fixed at D in its position of equilibrium. The removal of the part DA would not affect the curve assumed by DOB. This applies of course to both parabola and catenary.

The algebraic difficulties due to the transcendental equation (3) can often be avoided by expanding the exponential terms into series. Where c is large, that is, the horizontal tension great as compared with the weight per unit length, the series are sufficiently convergent to be useful. We find them to be

$$e^{\frac{x}{c}} = 1 + \frac{x}{c} + \frac{x^2}{2\,c^2} + \frac{x^3}{6\,c^3} + \text{etc.}$$

$$e^{-\frac{x}{c}} = 1 - \frac{x}{c} + \frac{x^2}{2\,c^2} - \frac{x^3}{6\,c^3} + \text{etc.}$$

The approximate value of y in terms of x is

$$y - c = \frac{x^2}{2\,c}. \tag{4}$$

To this degree of accuracy, the parabola replaces the catenary. It can be seen directly, that in a taut rope the difference between arc and abscissa becomes small.

CHAPTER IX

DIMENSIONS. THE WEIGHT-SYSTEM

142. In the preceding chapters we have used measurements according to the centimeter-gram-second scheme exclusively, postponing the assignment of explicit reasons for employing it. But the ends that such a scheme aims to secure should be definitely recognized and connectedly stated. The C.G.S. measures of physical quantity embody two distinct forms of advantage: one incidental and on the side of utility, while the second is a vital matter of theoretical adaptation; we can dismiss the former with brief mention. The C.G.S. plan is compact and serviceable, because the multiples and sub-multiples of its units fit the decimal notation; and, further, computation is simplified by systematized relations among its various units, awkward proportional factors being avoided. For example, the dyne involves unit mass and unit acceleration, the erg is a dyne-centimeter, and the watt is a volt-ampere. Yet the strongest claim of the C.G.S. system upon acceptance in scientific work is founded upon its theory. It is adopted because it enables us to express other physical quantities in terms of its three fundamental standards that are independent of local conditions. As a result of suggestion by Gauss, several schemes of this nature were proposed, in order to render all physical measurements directly comparable, and to that extent objective. The C.G.S. system is the only one that has really survived among these plans for so-called absolute measurement. The word "absolute" is here used in a special sense; it does

not indicate complete freedom from arbitrary conventional standards, but emphasizes the elimination of dependence upon the time and place at which observations are made. It can be perceived readily that we gain by minimizing the number of arbitrary standards needing to be preserved. And in the first proposal the earth itself was to furnish elements determining the mean solar second and the centimeter, the gram being defined in terms of water, a substance obtainable in purity everywhere. For practical reasons, however, that ideal has been relinquished in favor of the arbitrary definitions of the gram and centimeter that are now current.

The central purpose here being to establish a group of units that are consistently related to each other through the three independent standards, the particular phase of such connection for each unit enters essentially into our thought regarding it. This is exhibited in detail as a statement of its **dimensions**. In the technical use of the term, any quantity is said to be of n dimensions in respect to another, when the unit of the former varies as the nth power of the unit for the latter; absence of connection between the two quantities being expressed as zero dimension. Thus kinetic energy is of the first dimension as regards mass, and the second as regards speed; angle, with all other *ratios*, is of zero dimension in mass, length, and time. But special importance attaches to relations between the fundamental standards and units derived from them; that is, expressed in terms of them. The dimensional formula may be viewed as an accompaniment of each algebraic value that is not affected by changes in magnitude of the units involved; it presents a generalized aspect of interdependence among quantities. For this purpose the distinction between scalar and vector disappears, and the operation of projection is without influence.

Terms thus completely written consist of a magnitude-factor,

called the **numeric** because its numerical value varies according to the units selected, and the permanent dimensional factors. We shall write the latter $[M]$, $[L]$, $[T]$, with the proper exponents to indicate how the standards for mass, length, and time, respectively, enter into the units for the quantities in question. The dimensional formulas are appended for all the quantities that have become prominent thus far; they are verified without difficulty if we remember that each quantity carries into an expression the specification of its unit, as well as its magnitude. The volume abc of a rectangular prism is $a\,[\text{cm.}] \times b\,[\text{cm.}] \times c[\text{cm.}] = abc\,[\text{cm.}]^3$.

Speed, velocity,	$[M]^0[L]\,[T]^{-1}$
Angular velocity,	$[M]^0[L]^0[T]^{-1}$
Acceleration,	$[M]^0[L]\,[T]^{-2}$
Angular acceleration,	$[M]^0[L]^0[T]^{-2}$
Force,	$[M]\,[L]\,[T]^{-2}$
Impulse, momentum,	$[M]\,[L]\,[T]^{-1}$
Work, kinetic energy,	$[M]\,[L]^2[T]^{-2}$
Force-moment,	$[M]\,[L]^2[T]^{-2}$
Moment of inertia,	$[M]\,[L]^2[T]^0$
Moment of momentum,	$[M]\,[L]^2[T]^{-1}$
Power,	$[M]\,[L]^2[T]^{-3}$

When used without the numerical or algebraic factors, a **dimensional equation** may furnish a direct and valuable test of correctness in operations. All the terms of an equation must be homogeneous in their dimensions, for we add and equate quantities of the same kind. As an example, analyze equation (1), § 78, in this way; we find

$$([M]\,[L]\,[T]^{-2})\,[L] + [M]\,[L]^2\,[T]^{-2} = [M]\,([L]\,[T]^{-1})^2 + ([M]\,[L]^2)\,([T]^{-1})^2.$$

It is evident that the dimensional results are obtained by the ordinary rules for exponents; the terms are all of the order $[M][L]^2[T]^{-2}$.

143. It is sometimes necessary to clear physical thought that the element of dimensions be kept in mind as a distinction between numerically or algebraically equal measures of different quantities. To take a simple instance, the length traversed *in one second* at uniform speed is numerically equal to the length *per second* which measures the speed. But the dimensional relation between distance and speed is $([L][T]^{-1})[T]=[L]$, giving only numerical identity when the second time-factor is unity. In order to avoid confusion in such cases, the word "per" has been used consistently in the text to denote that the quantity following it occurs as a divisor, and enters into the dimensional formulas accordingly. By paying attention to this point, it will be observed that homogeneousness is maintained in the dimensions, where to superficial consideration it might seem to be ignored. Note as examples the factors k_1 (§ 118), q (§ 129), ϵ (§ 141). The definitions of field and potential (§§ 102, 113, 114) are framed in conformity with the same usage. The standard view of these conceptions specifies them as force and work *per* unit (in the body affected) of the measured quality that is subject to the influence of the particular field. They are not force and work *upon* a unit of the corresponding quantity exposed to the field, the unit-factors being suppressed.

Let us examine the fundamental expression for the law of gravitation as a dimensional equation, in close connection with the preceding paragraph. Using $[\theta]$ to signify the dimensions involved in θ, we have, since U is force (§ 102),

$$[M][L][T]^{-2}=[\theta][M]^2[L]^{-2};\quad [\theta]=[M]^{-1}[L]^3[T]^{-2}. \quad (1)$$

The numerical value of θ can be changed from 666×10^{-10} to unity by choosing as unit mass 3875 grams instead of one gram; but the order of θ as determined from the dimensions in (1) cannot be altered while mass (measured inertia), length, and time are preserved as independent standards.

If two homogeneous spheres of 3875 grams mass each could be placed with their centres one centimeter apart, their mutual attraction due to gravitation would be one dyne; for in magnitude

$$P = 666 \times 10^{-10} \cdot \frac{(3875)^2}{1^2} = 1. \qquad (2)$$

This should be compared with the statement defining unit of magnetic pole-strength, for the purpose of noting dimensional difference between the latter quantity and mass. We are at liberty so to choose the factors μ and μ_1, belonging to two magnets and determining the stress between two poles, that

$$P = k\frac{\mu\mu_1}{r^2}. \qquad (3)$$

Here the poles are supposed located at a distance r apart, and k is a proportional factor without dimensions. Then the necessary dimensions of μ or μ_1 are discoverable from the relation

$$[M][L][T]^{-2} = [\mu]^2[L]^{-2}; \quad [\mu] = [M]^{\frac{1}{2}}[L]^{\frac{3}{2}}[T]^{-1}. \qquad (4)$$

We may pursue this subject one step farther, so as to place in parallel the dimensions of gravitational and magnetic field. The former is force per unit mass,

$$\frac{P}{m} = \frac{[M][L][T]^{-2}}{[M]} = [L][T]^{-2}\,\text{(Acceleration).} \qquad (5)$$

Magnetic field is force per unit of pole-strength,

$$\frac{P}{\mu} = \frac{[M][L][T]^{-2}}{[M]^{\frac{1}{2}}[L]^{\frac{3}{2}}[T]^{-1}} = [M]^{\frac{1}{2}}[L]^{-\frac{1}{2}}[T]^{-1}. \qquad (6)$$

This should serve as a corrective to the idea that the quantity called field is of the same order in all cases where force follows the law of inverse square.

144. Though the C. G. S. system is so excellently contrived, it has not been universally adopted. In certain problems of engineering, weight is an important factor in the form of load to be supported or resistance to be overcome. Consequently, here as in the affairs of common life, the older practice is still followed, that finds in weight a standard with which to compare other forces. This is the characteristic of the **weight-system**, now to be described in application, first to English and then to metric measures.

For everyday purposes, the foot and the pound may be taken as the units of length and mass respectively; the centimeter and gram being replaced by these other arbitrary units, while the mean solar second is retained. Velocity is then systematically expressed in feet per second $\left(\frac{\text{ft.}}{\text{sec.}}\right)$; and acceleration in [feet per second] per second $\left(\frac{\text{ft.}}{\text{sec.}^2}\right)$. But the weight-system departs from the C. G. S. model at this point, and adds, by a fourth convention, as force-unit, the **pound-weight**; *i.e.* the weight of a one-pound mass. Since the weight of a given mass is not a fixed quantity at different places on the earth's surface, a definite locality is next assigned, in order to avoid a variable or local unit, and make the measurements absolute in the sense explained above (§ 142). The conditions usually selected are the sea-level at latitude 45° N., the standard value of g being $32.173\ \frac{\text{ft.}}{\text{sec.}^2}$ or $980.6\ \frac{\text{cm.}}{\text{sec.}^2}$. We shall denote this particular weight-constant by g_0; it is of course reckoned for a vacuum.

Force is in general proportional to the product of mass and acceleration; it remains to evaluate the constant factor with the present choice of units. Let a pound-mass be given under

the standard conditions; then the force exerted upon it by the earth is one pound-weight, and the equation of motion $P=kmp$ becomes $1=k\times 1\times g_0$, giving $k=\frac{1}{g_0}$. Hence the form of such equations adapted to the weight-system and English measure is

$$P(\text{lbs.-wt.})=\frac{1}{g_0}\cdot m\,(\text{lbs.})\cdot p\left(\frac{\text{ft.}}{\text{sec.}^2}\right). \qquad (1)$$

The constant factor $\frac{1}{g_0}$ is of zero dimensions in mass, length, and time, being determined as a ratio.

If for pound we substitute gram or kilogram, and for foot centimeter or meter, we obtain by inspection alternative forms applying to the metric system, the force-unit being always the weight (under the same standard conditions) of the mass-unit,

$$P(\text{grams-wt.})=\frac{1}{g_0}\cdot m\,(\text{grams})\cdot p\left(\frac{\text{cm.}}{\text{sec.}^2}\right) \qquad [g_0=980.6]. \quad (2)$$

$$P(\text{kg.-wt.})=\frac{1}{g_0}\cdot m\,(\text{kg.})\cdot p\left(\frac{\text{meters}}{\text{sec.}^2}\right) \qquad [g_0=9.806]. \quad (3)$$

Therefore the one algebraic formula summarizes the equation of motion for the weight-system in all these forms,

$$P=\frac{1}{g_0}mp. \qquad (4)$$

When we are dealing with weight, $p=g$, the local value of the weight-constant, and the general expression becomes $P_w=\frac{mg}{g_0}$. The numerical value of $\frac{g}{g_0}$ usually differs from unity by less than one-fourth of one per cent in actual cases, an amount that can be disregarded in the ordinary practice to which the weight-system would be applied. Hence the weight (P_w) and the mass (m) can be treated as numerically equal, with the convenient result that the former is then given directly by the operation of weighing.

The modification begun in the equation of motion must be continued into the equations for impulse, work, and power. But further discussion can be attached to (4), since it is now apparent that any difference between English and metric systems lies entirely in numerical values. Elementary combinations will exhibit the changes to be made; therefore let equation (4) be applied to tangential force and rectilinear translation. Its integral with respect to time gives the impulse equation

$$\int_{t_0}^{t} P_t\, dt = \frac{m(v - v_0)}{g_0}. \tag{5}$$

By considering momentum as a quantity *proportional* to the product of mass and velocity, the type being $\frac{mv}{g_0}$ in the present instance, equation (5) is kept in harmony with the verbal statement of § 53. The systematic simplicity of the C.G.S. system having been abandoned in the measure of force, the same proportionality replaces equality at other points.

Holding to the simplified conditions, and proceeding to the work equation, we find

$$\int_{s_0}^{s} P_t\, ds = \frac{m(v^2 - v_0^2)}{2\, g_0}. \tag{6}$$

Regard kinetic energy as *proportional* to the product of mass and the half-square of speed, and let $\frac{mv^2}{2\,g_0}$ represent that quantity typically in the weight-system. Then the verbal statement of § 57 concerning the relation between work and kinetic energy is made to cover equation (6). The work-unit for English measures is defined as the work of one pound-weight for a displacement of one foot in its own direction, and designated by common usage as the **foot-pound**. The corresponding metric work-units are the **gram-centimeter** and the **kilogram-meter**, standardized as suggested by the terms.

It is clear, of course, that the adoption of the weight-system involves the same modification (division by g_0) running through all expressions into which kinetic energy and momentum enter. For example, the moment of momentum of a rotating body becomes $\frac{\omega}{g_0} I_0$; its kinetic energy appears in the form $\frac{\omega^2}{2g_0} I_0$.

The power-unit of the weight-system is the **horse-power.** In connection with English measures, this is specified as a work-rate of 550 foot-pounds per second; as a metric quantity it is 75 kilogram-meters per second. The latter has been made a convenient round number, and thrown out of exact correspondence with the English horse-power.

In speaking of units as fundamental or derived, we do not imply that the former are more directly related to experience. The terms refer rather to a reconstructed mathematical order; those units being fundamental that are defined first, and upon which others are made to depend. The weight-system probably stands closer in two respects to the instinctive sequence of ideas from which science has been obliged to emancipate itself. First, in placing weight at the front, as the type of all force; and, secondly, in subordinating, more or less consciously, the conception of mass. The endeavor has been made in the text to put the leading thought of the weight-system in such form that it is thoroughly scientific. Thus restated, it is shown as an absolute system with four conventional standards instead of three; and as a consequence, with a proportional factor differing from unity in the fundamental equations. By comparison, it brings into relief the greater simplicity for most purposes of the C. G. S. scheme; but there is no reason why engineers should not use the weight-system, if as a matter of fact it is better suited to some of their needs.

CHAPTER X

CENTRE OF MASS. MOMENT OF INERTIA

145. The standard descriptions of motion in dynamics give prominent place to a point known as **centre of mass** and defined by the following condition: For any given mass-distribution, the centre of mass is that point whose distance from any plane is the average of the corresponding coördinates for all elements of mass in the body or system. Note particularly, first, that the average is not to be formed on the basis of (geometrical) volume, but with regard to the distribution of mass; and, secondly, that the distance is measured from a plane, not from a line or point. For purposes connected with the dynamics of rigid bodies, the assumption is made that mass is a continuous function of the coördinates within the boundaries of a single body. This is done in order that the volume assigned to it may be covered by processes of summation for quantities like mass, force, energy, in the convenient form of integration. But in dealing with a system of rigid bodies, spaces where there is no mass may occur between them. In order to include summation of dynamical quantities throughout a system, therefore, the sign (Σ) must in general be combined with the sign of integration $\left(\int\right)$. For brevity we shall adopt the latter symbol exclusively, with the understanding that it includes the double process, if circumstances render both necessary. The letter C will commonly denote the centre of mass, and the mark ($\bar{\ }$) is placed over its coördinates.

Let x be the coördinate measured from a plane YZ to any differential mass dm. In accordance with the conception of an average we can write, m_1 being the entire mass included,

$$m_1\bar{x} = \bar{x}\int_0^{m_1} dm = \int_0^{m_1} x\,dm; \quad \bar{x} = \frac{\int_0^{m_1} x\,dm}{\int_0^{m_1} dm} = \frac{\int_0^{m_1} x\,dm}{m_1}. \quad (1)$$

This equation applies whether x be constant, or variable with regard to time, *i.e.* whether the plane and the system are relatively in motion or at rest. The distance $\bar{x}$ may have an instantaneous or a permanent value. Similar equations can be obtained for the two other coördinates by changing the variable, giving

$$\left.\begin{aligned} m_1\bar{y} &= \int_0^{m_1} y\,dm; \quad \bar{y} = \frac{\int_0^{m_1} y\,dm}{m_1}; \\ m_1\bar{z} &= \int_0^{m_1} z\,dm; \quad \bar{z} = \frac{\int_0^{m_1} z\,dm}{m_1}. \end{aligned}\right\} \quad (2)$$

The coördinates $\bar{x}$, $\bar{y}$, $\bar{z}$ locate the centre of mass relatively to the rectangular axes X, Y, Z.

146. Let the planes YZ and XZ be turned about the Z axis through the angle α; then the coördinates of dm at (x, y) become x_1, y_1. The equations connecting the two pairs of coördinates are

$$x = x_1 \cos\alpha - y_1 \sin\alpha; \quad y = y_1 \cos\alpha + x_1 \sin\alpha. \quad (1)$$

Substituting these values in (1) and (2), § 145, the result is, α being a common factor for all mass-elements,

$$m_1\bar{x} = \cos\alpha\int_0^{m_1} x_1\,dm - \sin\alpha\int_0^{m_1} y_1\,dm = m_1\bar{x}_1\cos\alpha - m_1\bar{y}_1\sin\alpha; \quad (2)$$

$$m_1\bar{y} = \cos\alpha\int_0^{m_1} y_1\,dm + \sin\alpha\int_0^{m_1} x_1\,dm = m_1\bar{y}_1\cos\alpha + m_1\bar{x}_1\sin\alpha. \quad (3)$$

On dividing by m_1 we see that the condition is fulfilled that gives the point $(\bar{x}_1, \bar{y}_1)$ the same position relative to the system as the point $(\bar{x}, \bar{y})$, the coördinate $\bar{z}$ being unchanged. A similar process can be carried through for all the planes and axes; so we conclude that the choice of particular axes at a given origin does not influence the location of C relatively to the system of bodies of which it is the centre of mass.

Neither does a change of origin to the point (x_0, y_0, z_0), the planes remaining parallel to their previous positions, affect the location of C with respect to its own system or body. Placing $x = x_0 + x_1$, substitution in equation (1), § 145, yields

$$m_1\bar{x} = \int_0^{m_1} x_0\, dm + \int_0^{m_1} x_1\, dm = m_1 x_0 + m_1\bar{x}_1. \tag{4}$$

Dividing by m_1, the announced result is at once apparent for X, and similar equations for Y and Z can be written by inspection. Consequently the centre of mass of any body or system is a point definitely related to it at each instant, with complete independence of the particular set of coördinate planes by means of which its situation happens to be determined. As a matter of practical detail in calculation, the coördinate planes are generally assumed to be fixed relatively to some body in the system. The centre of mass in a given case may of course be at a place free from mass (for example, in a hollow space). When a body is rigid, C accompanies it as though rigidly attached.

Under least favorable conditions, all three coördinates of C must be calculated by processes involving integration. The mathematical difficulties are notably reduced, however, in many actual cases, when a body is: (1) homogeneous; and (2) symmetrical. The first specification implies that the density δ has a common value at all parts of the mass, giving $dm = \delta\, dV$

for *any* element of volume dV (*i.e.* with the same value of δ). The integrals to be evaluated fall under the type

$$\delta V\bar{x} = \delta\int_0^{V_1} x\,dV; \tag{5}$$

and the result is geometrical, or independent of the material. The centre of mass becomes identical with the centre of figure or centroid. In addition, now let the body be symmetrical with respect to one plane; then C must lie in that plane. For we may measure x as distance from the plane of symmetry; and in the integral $\int_0^{m_1} x\,dm$ two equal elements of mass (δ being common) have coördinates of equal length $\pm x$. The terms included in the integral are self-cancelling pairs; hence

$$m_1\bar{x} = \int_0^{m_1} x\,dm = 0; \quad \bar{x} = 0. \tag{6}$$

We see further that when there are two planes of symmetry C must lie in their line of intersection; and that the point is completely determined in position if the body has three such planes not intersecting in the same line.

The problem of finding the position of the centre of mass in a homogeneous cone affords a sufficient example of the general method. The point C must be situated in the cone-axis, as a result of symmetry; it remains only to locate it in that line. Pass the plane YZ through the vertex and assume X along the axis. The work can be shortened as a rule (always with possible exceptions) by choosing the differential mass as large as permissible; but x must have a common value at all its parts. In this instance, let dm be the mass of a circular section distant x from the vertex, and of thickness dx. Denote the semi-angle of the cone by α, its density by δ, and its altitude by x_1. The equation to determine $\bar{x}$ becomes

$$\delta\cdot\pi x_1^2\,\mathrm{tg}^2\,\alpha\cdot\frac{x_1}{3}\cdot\bar{x} = \int_0^{x_1}\delta\cdot\pi x^2\,\mathrm{tg}^2\,\alpha\cdot dx\cdot x. \tag{7}$$

Dividing by the common constant factors and integrating, we find

$$\frac{x_1^3}{3}\cdot\bar{x}=\frac{x_1^4}{4};\quad \bar{x}=\tfrac{3}{4}x_1. \tag{8}$$

147. Every final average may be arrived at by successive steps, and expressed as the average of partial averages. This idea justifies the equation

$$\int_0^m x\,dm = m\bar{x} = m_1\bar{x}_1 + m_2\bar{x}_2 + m_3\bar{x}_3, \tag{1}$$

if $m = m_1 + m_2 + m_3$, and $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$ are partial average values, each for the mass by which it is multiplied. This form of statement suggests several useful consequences. First, it covers the case of a system, the masses m_1, m_2, m_3 being separated by spaces without mass. Locate their centres of mass for the bodies separately; then determine the common centre for the system by equation (1). Secondly, it may happen that a body has not the cross-section of a regular solid whose area is a simple function of its normal coördinate; and yet the body may be built up out of parts that are thus tractable. Equation (1) is serviceable in such instances, and it applies whether the constituent parts have the same density or not. Thirdly, one or more terms may be transferred to the first member. Under that form the equation enables us to calculate the position of the centre of mass for a regular solid in which there are regular cavities. Write equation (1)

$$\frac{m\bar{x}-m_3\bar{x}_3}{m-m_3}=\frac{m_1\bar{x}_1+m_2\bar{x}_2}{m_1+m_2}\quad [m-m_3=m_1+m_2]. \tag{2}$$

Regard the body as brought to the mass m by successive stages; then either member of (2) gives the average coördinate just before m_3 is added. In other words, the centre of mass is determined while there is a cavity with (mass) contents m_3 in a solid which when complete has mass m.

For certain purposes, chiefly those of generalized proof, the equations of type like (1), § 145, are written as triple integrals of a volume element. They appear under those circumstances in the form, δ being variable,

$$\bar{x} = \frac{\int\int\int \delta x \, dx \, dy \, dz}{\int\int\int \delta \, dx \, dy \, dz}. \tag{3}$$

148. There is one type of integral which recurs in so many important dynamical investigations that a group of general propositions concerning it has accumulated. These may not be strictly a part of dynamics; but their connection with it through use and interpretation is so close that it is natural to establish and discuss them under the heading of Mechanics rather than mathematics. The integral in question sums terms that are the product of each differential mass by the square of its distance from the same line or axis, for the entire mass of a rigid body. Every element in integrals of this kind is then essentially positive. By fixed usage, the term applied to such an integral is **moment of inertia**; the quantity expressed by it belongs to a given rigid body in determinate relation to one particular line; in cases of pure rotation about the line as an axis it is therefore constant for all values of time, and of angular displacement. We shall denote moment of inertia by the letter I, with a subscript that indicates the axis with reference to which it is calculated. Let r be the distance of any differential mass dm from the axis X, and m_1 the mass of the (rigid) body; then in symbols

$$I_X = \int_0^{m_1} r^2 \, dm = \int_0^{m_1} (y^2 + z^2) \, dm. \tag{1}$$

The routine of evaluating these integrals will be understood readily from the examples that follow; it is seen to be similar

to the process explained in connection with centre of mass. We shall first calculate the moment of inertia for a *homogeneous* cylinder, with respect to its geometrical axis. As in the parallel instance, § 146, select the differential element as large as is allowable under the necessary condition that its parts shall be equally distant from the axis (the value of r must be common to dm); the plain suggestion here is to take concentric cylindrical shells. Let δ be density, h altitude, r_1 radius, m_1 mass, of the cylinder; r is the radius of a shell, and dr its thickness. Calling the moment of inertia I_A, we have

$$I_A = \int_0^{r_1} r^2(\delta \cdot h \cdot 2\,\pi r\,dr) = \delta\pi h\frac{r_1^4}{2} = m_1\frac{r_1^2}{2}. \tag{2}$$

The moment of inertia of a homogeneous sphere with reference to a diameter is another value that is frequently called for. Conceive the sphere as composed of disks with differential thickness, their centres lying on the chosen diameter, and their planes being perpendicular to it. Let δ be the density, m_1 the mass, and r_1 the radius, of the sphere; r the radius of a disk, x its distance from the centre of the sphere, and dx its thickness. The equation connecting r and x is $r^2 + x^2 = r_1^2$. If I_D is the moment of inertia required, we have, using for each disk the previous result,

$$\begin{aligned} I_D &= 2\int_0^{r_1}\delta \cdot \pi r^2\,dx \cdot \frac{r^2}{2} = \pi\delta\int_0^{r_1} r^4\,dx \\ &= \pi\delta\int_0^{r_1}(r_1^4 + x^4 - 2\,r_1^2x^2)\,dx = \tfrac{8}{15}\,\pi\delta r_1^5 = \tfrac{2}{5}\,m_1r_1^2. \end{aligned} \tag{3}$$

149. The moment of inertia desired may sometimes be derived conveniently as the sum of partial moments of inertia, the idea being formally parallel to that of equation (1), § 147, in so far as the quantities are all definite integrals. Thus

$$\int_0^{m} r^2\,dm = \int_0^{m_1} r^2\,dm + \int_{m_1}^{m_2} r^2\,dm + \text{etc.} \tag{1}$$

The conception of a given mass-distribution as arising by partial removal of mass from a larger body is also useful. Limiting the second member of (1) to two terms, m_2 becomes m. On transposing its first term, we obtain

$$\int_0^m r^2\,dm - \int_0^{m_1} r^2\,dm = \int_{m_1}^m r^2\,dm. \tag{2}$$

By way of illustration, let us apply the idea to the case of a ring, whose section by a plane through its axis is two rectangles. Consider this as the part remaining of a solid homogeneous cylinder of mass m after removing a concentric core of mass m_1, and express the moment of inertia for their common axis. We find when the ring has inside radius r_1, and outside radius r_2,

$$I_A = m\frac{r_2^2}{2} - m_1\frac{r_1^2}{2}. \tag{3}$$

In terms of dimensions and density, $m = \delta\pi r_2^2 h$; $m_1 = \delta\pi r_1^2 h$;

$$I_A = \frac{\delta\pi r_2^4 h}{2} - \frac{\delta\pi r_1^4 h}{2} = m_2\frac{r_2^2 + r_1^2}{2} \qquad [m_2 = m - m_1]. \tag{4}$$

150. We shall next prove the useful relation that is found to exist among the moments of inertia of a body for all parallel axes. Let the diagram (Fig. 39) represent the section of any given body which contains its centre of mass C, and is perpendicular to any group of parallel axes. Take any other point in the section as O, assume CO as X, and origin at C; the YZ plane is shown in the diagram by its trace. Let $CO = \bar{r}$, and I_C, I_O be moments of inertia for the axes of the group which pass through C and O respectively. Regard the body as composed of prismatic elements parallel to those axes; and select any element, shown at A; its distance from C is r_1, and from O is r. With this notation we can indicate

$$I_O = \int_0^{m_1} r^2\,dm; \quad I_C = \int_0^{m_1} r_1^2\,dm. \tag{1}$$

In the triangle CAO, the angle at C being α, we have

$$r^2 = r_1^2 + \bar{r}^2 - 2\,\bar{r}r_1 \cos\alpha = r_1^2 + \bar{r}^2 - 2\,\bar{r}x. \tag{2}$$

By substitution in the value of I_O, this gives

$$I_O = \int_0^{m_1} r_1^2\,dm + \bar{r}^2 \int_0^{m_1} dm - 2\,\bar{r} \int_0^{m_1} x\,dm. \tag{3}$$

But as regards the last integral we see $\int_0^{m_1} x\,dm = m_1\bar{x} = 0$, because the origin is at C. Hence finally

$$I_O = I_C + m_1\bar{r}^2. \tag{4}$$

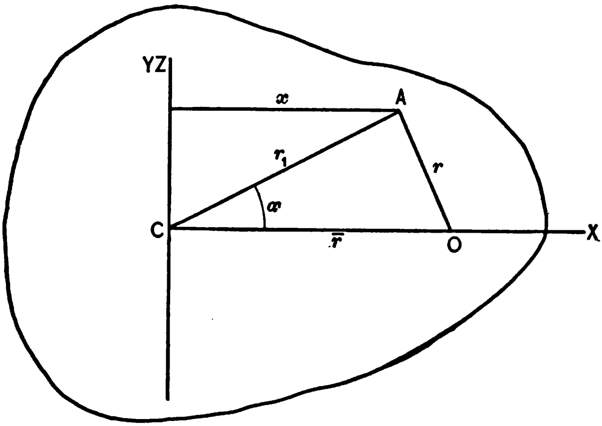


FIG. 39

Write a similar equation for a parallel axis at any other point Q in the section, whose distance from C is $\bar{r}_1$,

$$I_Q = I_C + m_1\bar{r}_1^2. \tag{5}$$

On subtracting (5) from (4) the relation between the moments of inertia for parallel lines through Q and O appears,

$$I_O = I_Q + m_1(\bar{r}^2 - \bar{r}_1^2). \tag{6}$$

Consequently the moment of inertia has equal values for the generators of any cylindrical surface whose axis passes through

the centre of mass. In view of the result in equation (4), the centre of mass for any rigid body is seen to possess this remarkable property: Among the moments of inertia calculated for any group of parallel lines, that for the line passing through the centre of mass is an absolute minimum.

An auxiliary proposition of narrower scope is the following: The moment of inertia for any line normal to a lamina *whose thickness is negligible* is the sum of the moments of inertia for any rectangular axes in its plane, and intersecting the normal axis. Take Z normal to the plane; then

$$I_X=\int_0^{m_1} y^2\,dm;\quad I_Y=\int_0^{m_1} x^2\,dm;\; I_Z=\int_0^{m_1}(x^2+y^2)\,dm=I_X+I_Y. \tag{7}$$

The results proved above shall be applied in calculating the moment of inertia of a homogeneous cylinder for a diameter of its median (circular) section. Let its density be δ, radius r_1, altitude $2h$; then its mass $m_1=2\,\delta\pi r_1^2 h$. Regard a circular lamina of thickness dx, whose plane is distant x from C, the middle point of the cylinder-axis. Its moment of inertia for any of its own diameters is $dm\dfrac{r_1^2}{4}=\delta\pi r_1^2 dx\cdot\dfrac{r_1^2}{4}$ (eq. (7)). Transferring to a parallel diameter of the median section adds the term $dm\,x^2$ (eq. (4)). Summing such elements for the whole cylinder,

$$I_D=2\int_0^h \delta\pi r_1^2\left(\frac{r_1^2}{4}+x^2\right)dx=\delta\pi r_1^2 h\left[\frac{r_1^2}{2}+\frac{2}{3}h^2\right]=m_1\left[\frac{r_1^2}{4}+\frac{h^2}{3}\right]. \tag{8}$$

When the density is a function of position in the body, and not constant as we have thus far considered it in our calculations, the more general expression must be taken,

$$I=\iiint r^2\delta\,dx\,dy\,dz. \tag{9}$$

151. The main proposition of the preceding section (eq. (4)) enables us to ascertain the moment of inertia for a given body

and axis, when its value for a parallel line through the centre of mass and the distance of this point from the first axis are known. In this way lines at the centre of mass furnish standards of reference for transfer to parallel axes. But, by proper selection of three standard lines, moment of inertia for any line through the centre of the mass can be expressed in terms

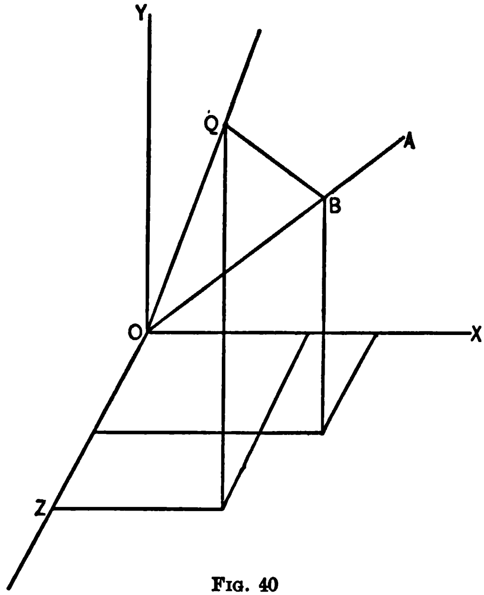


FIG. 40

of the same quantity for rectangular axes at that point. It is our next undertaking to establish this as a general proposition, and to show what elements determine the choice of such standard directions.

Let XYZ (Fig. 40) be rectangular axes, with origin at any point O, and OA any line drawn through the origin, l, m, n being its direction-cosines. We shall denote moment of inertia for OA by I_A, and proceed to express it as a function of

I_X, I_Y, I_Z, together with such other quantities as may appear necessary. Let Q at (x, y, z) represent any differential mass dm of the body for which I_A is to be determined; and denote BQ, its distance from OA, by r_1. The geometrical relations are

$$r_1^2 = \overline{OQ}^2 - \overline{OB}^2 = x^2 + y^2 + z^2 - (lx + my + nz)^2. \qquad (1)$$

Since $l^2 + m^2 + n^2 = 1$, we can write $x^2 = (l^2 + m^2 + n^2)\,x^2$, and similarly for y^2 and z^2. Using this device, expanding the square of the trinomial, and rearranging terms, we obtain

$$\begin{aligned} r_1^2 &= l^2(y^2 + z^2) + m^2(z^2 + x^2) + n^2(x^2 + y^2) \\ &\quad - 2\,mn\,yz - 2\,nl\,zx - 2\,lm\,xy. \end{aligned} \qquad (2)$$

Hence, forming the mass-integral, the resulting equation is

$$\begin{aligned} I_A = \int_0^{m_1} r_1^2\,dm &= l^2\int_0^{m_1}(y^2 + z^2)\,dm + m^2\int_0^{m_1}(z^2 + x^2)\,dm \\ &\quad + n^2\int_0^{m_1}(x^2 + y^2)\,dm - 2\,mn\int_0^{m_1} yz\,dm \\ &\quad - 2\,nl\int_0^{m_1} zx\,dm - 2\,lm\int_0^{m_1} xy\,dm. \end{aligned} \qquad (3)$$

The last three integrals are known as products of inertia. Designating them as D, E, F, in that order, we may write

$$I_A = l^2 I_X + m^2 I_Y + n^2 I_Z - 2\,mn\,D - 2\,nl\,E - 2\,lm\,F. \qquad (4)$$

In order to express moment of inertia for OA in terms of moments of inertia for the axes alone, we must be able to determine X, Y, Z so that $D = E = F = 0$. When we are concerned with regular (and homogeneous) solids, an inspection of their symmetry is often sufficient to show how the axes can be so chosen as to fulfil that condition. For example, E and F vanish if YZ is a plane of symmetry, and similar conclusions can be drawn for the other planes. Axes with respect to which all three products of inertia become zero form a set of **principal axes** for their origin.

152. Let OA move as a radius-vector r about the origin as pole, its length being connected with the moment of inertia for the line with which it coincides, in the fixed relation: The square of the radius-vector is inversely proportional to the moment of inertia. The locus will be a closed surface, since no moment of inertia of a rigid body can be zero or infinite for any line through O. The condition under which the locus is described gives

$$I_A = \frac{a}{r^2} = l^2 I_X + m^2 I_Y + n^2 I_Z - 2\,mn\,D - 2\,nl\,E - 2\,lm\,F. \quad (1)$$

But the direction-cosines of r are l, m, n; hence for any point (x, y, z) of the surface, $x = lr$, $y = mr$, $z = nr$. Clearing equation (1) of fractions, and substituting as above, we find

$$a = I_X x^2 + I_Y y^2 + I_Z z^2 - 2\,Dyz - 2\,Ezx - 2\,Fxy. \quad (2)$$

The locus is therefore a closed quadric, which must be the surface of an ellipsoid. The equation is central, and by reference to the axes of the ellipsoid it takes the form

$$a = Ax^2 + By^2 + Cz^2. \quad (3)$$

A, B, and C are the particular values that I_X, I_Y, I_Z assume when the coördinate axes are the axes of the ellipsoid; they are the principal moments of inertia for the point O. Dividing (3) by r^2, and reversing the previous substitutions that gave equation (2), the final relation appears,

$$I_A = l^2 A + m^2 B + n^2 C. \quad (4)$$

Consequently, apart from the existence of symmetry, there are three principal axes at right angles for each point, determined as the axes of the ellipsoidal locus, the conditions for whose description about every centre are always fulfilled. It is self-evident, from the nature of the locus, that it is in determinate relation to the body at each point, and independent of the particular axes used in expressing equation (2); but it

may, of course, be changed in scale by varying the constant a. It is known as the momental ellipsoid. One of its axes is the longest of its diameters, and another is the shortest; hence of the principal moments of inertia one is greatest, and one least, among the values for axes intersecting at any origin. Reasons are assigned at the beginning of § 151, why the principal moments at the centre of mass are especially useful.

If two axes of an ellipsoid are equal, it becomes an ellipsoid of revolution. Its equation then retains the form (3) for any rectangular diameters of the circular section through its centre. Therefore all diameters of that section are principal axes at the centre, with equal moments of inertia. Conversely, if two principal moments of inertia at any point are known to be equal, the ellipsoid there is one of revolution. Further, if three principal moments are equal, the ellipsoid becomes a sphere, the moments of inertia are equal for all axes through its centre, and all diameters are principal axes at the centre.

The value of the ellipsoid is found chiefly in general conclusions like those just mentioned and in methods of statement which it renders possible by introducing principal axes into discussions of dynamical theory. We seldom need to locate principal axes at a given point by calculation based on the equation of the locus for that origin.

GENERAL EXERCISES AND PROBLEMS

[NOTE. The remarks with which Chapter VIII begins apply here also. Especially, all bodies are to be treated as homogeneous and rigid, where the assumption of those qualities is not excluded by the context.]

1. A circle rolls upon a straight line at a uniform rate, and a point in its circumference describes a cycloid. What is the average speed, from cusp to cusp, with which the point moves?

2. Prove by kinematical considerations that the tangent and normal to a cycloid intersect the generating circle at the extremities of the diameter drawn through the point of contact with the line on which the circle is developed.

3. Show kinematically that the tangent at each point of an ellipse bisects one angle between lines drawn to the foci. Establish the corresponding proposition for a parabola.

4. What is the average speed between two successive positions of rest for any point of a piston rod connected with a crank that is turning uniformly?

5. Given that a point moves subject to the condition

$$v_x \cos\alpha + v_y \cos\beta + v_z \cos\gamma = 0,$$

with α, β, γ constant. Determine its locus.

6. Given a spherical locus; express the condition to which v_x, v_y, v_z are subject, when the origin is at its centre.

7. Wind is blowing steadily from N. $\alpha°$ W. Determine its velocity in magnitude and direction relatively to a train that is moving due north at a given rate. Required the rate at which the wind is veering for a person on the train, supposing that it moves from rest with constant acceleration.

8. A point P describes a circle with constant speed relatively to a plane that has a constant velocity $\widehat{v}$ of translation referred to the earth. Plot the velocities of P (relative to the earth) perpendicular and parallel to $\widehat{v}$ as functions of the time.

9. What is the equation of a path in describing which the resultant velocity makes a constant angle with the radius-vector?

10. Two points move so that the line joining them always passes through the same fixed point, and is divided by it in a constant ratio. What relation exists between the two paths?

11. A rectangular metal sheet has motion of translation between guides in contact with one pair of its sides. A pin attached to a crank that turns about a fixed axis normal to the sheet engages in a straight slot cut in the latter at a given angle with the guides. Discuss the motion of the sheet corresponding to uniform rotation of the crank.

12. A rod is moved in the direction of its length by a cam that turns about a normal axis perpendicular to the rod and intersected by its prolongation. The velocity curve of the rod as a function of time is given; determine the conditions to be fulfilled by the cam.

13. Treat the parabolic trajectory by the method of changing origin, using as intermediate origin a point moving with the initial velocity.

14. How can a point move in a curved path, subject to the condition that the resultant acceleration makes a constant angle (*a*) with the tangent? (*b*) with the radius-vector?

15. A point moves so that its radial acceleration and the angular velocity of its radius-vector are both constant. Deduce the value of the acceleration perpendicular to the radius-vector under these conditions.

16. If a circle is developed uniformly on a straight line, and a point in its circumference describes a cycloid, determine the radius of curvature at the vertex from the kinematical data.

17. Show how the radius of curvature in a parabolic trajectory may be calculated by using the values for velocity and acceleration.

18. A point describes a plane curve, and a perpendicular is let fall from any origin in its plane upon the instantaneous direction of the velocity v. Prove that the foot of this perpendicular moves at the rate $v\frac{r}{\rho}$, r being the radius-vector, and ρ the radius of curvature.

19. Prove analytically that the resultant of (p_x, p_y) has the same direction as the resultant of (p_t, p_n), or (p_r, p_γ).

20. Extend the scheme of polar components to three dimensions by letting the plane of the diagram (Fig. 6, p. 19) rotate at the required rate about Y. Deduce expressions for the component velocities and accelerations in terms of the polar variables r, γ, ϕ. The third variable is the angle at any instant between YOA and the position of that plane at the epoch $t = 0$.

21. A small mass falls freely along a vertical line intersecting the horizontal axis of a cylinder that turns uniformly. Required the curve of a channel to be cut in the cylinder so that the mass can just pass through without touching when it starts at a given distance above the cylinder-axis.

22. If a radius-vector BP moves about a fixed point B so as to represent continually the velocity of a moving point Q in direction and magnitude, the locus of P is called a hodograph. Show that the velocity of P represents the acceleration of Q. Prove that the hodograph is a curve of the same type as the path when Q describes a circle or a logarithmic

spiral with constant angular velocity about a pole. Examine whether the restriction last named is necessary.

23. Prove that the hodograph is a circle for any conic section described as a central orbit round a focus.

24. How must a point move in order that its hodograph may be a straight line and be traversed with constant speed?

25. A straight line PQ always passes through a given fixed point A, while P moves with constant acceleration along a fixed straight line CD. Express the resultant velocity and acceleration of Q at any position.

26. The line OP, rotating uniformly about an axis at O, is of constant length, and C is any fixed point in the plane of its motion. Q is a definite point in the line CQ, which moves about C so that it contains P always. Establish expressions for the velocity and acceleration of Q.

27. A point Q moves in a circle, and another point P in a tangent AP, each with constant tangential acceleration. They have zero velocity at the point of tangency initially. Express as functions of time the velocity and acceleration of Q referred to P as origin, and the tangent AP as reference-line.

28. Prove that the resultant acceleration of a point describing a cycloid may be represented by a radius of the generating circle, if the latter moves uniformly.

29. A cord is unwound at a constant rate from a circle, and kept taut. Required the path, velocity, and acceleration of its end referred to the centre of the circle and a definite diameter. Also the same elements referred to the instantaneous point of tangency and a definite tangent.

30. A circle of radius r rolls inside a fixed circle of radius R. Prove that the path of any point in the circumference of the moving circle is a straight line when $2r = R$. Express the

acceleration of the instantaneous centre, considered as a point of the rolling circle, for any value of the ratio $\frac{r}{R}$, the motion being uniform.

31. A circle of radius r has motion in its own plane describable as velocity of translation $\widehat{v}$ with its centre, and rotation ω about a normal axis at that point. Locate the instantaneous centre for any ratio of v and ω. Prove that the problem may be reduced to one of rolling along a straight line when $\widehat{v}$ and the ratio in question are constant.

32. A cylinder rocks about a diameter of one base, the angular motion being harmonic. A concentric cylindrical piston slides in and out, its translation relative to the cylinder being likewise harmonic with the same period and opposite phase. Determine the instantaneous centre for any point in the axis of the piston; also its resultant velocity and acceleration at any position.

33. Let lines AB, AC represent two vector quantities of the same kind to different scales, which are b and c for each unit of length in that order. Prove that the direction of the resultant is represented by AD, where D is so taken in BC that $b \cdot \overline{BD} = c \cdot \overline{CD}$. And that the magnitude of the resultant is $(b + c)\ \overline{AD}$. The ordinary parallelogram construction is included as a particular case if $b = c$.

34. Initial velocities v', v'', in the same vertical, are given to two small masses at an interval t. Determine the condition that the masses may meet, and the time and place of meeting in a possible case.

35. A mass having motion of translation from rest, with constant tangential acceleration, moves over a distance s_1 in the first t_1 seconds. Find the value of t when the distance $2\,s_1$ has been traversed.

36. Prove that speed in the parabolic path (§ 117) at any position is equal to that which would be acquired in falling freely to that level from rest on the directrix.

37. Let the speed of projection in a parabolic trajectory be known. Find the corresponding direction at a given origin, in order that the curve may pass through an assigned point (x_1, y_1). Discuss double solutions and impossible cases.

38. With known speed of projection, find the direction for maximum range in any azimuth on an inclined plane that passes through the initial position. Examine the statement that the focus of the parabolic trajectory lies in the inclined plane, when the condition for maximum range is fulfilled.

39. A small mass moves in a parabolic trajectory. Prove that its radius-vector drawn from the focus moves with angular velocity inversely proportional to its length.

40. At $t = 0$ a small mass is allowed to fall from that point in a vertical line at which it is intersected by the direction of the initial velocity for another mass started in a parabolic trajectory at the same instant. Prove that the two masses will meet.

41. Small masses are projected simultaneously from the same place with the same speed. Assuming that they describe parabolic trajectories, show that their locus at any subsequent epoch is a spherical surface.

42. Discuss the modification produced in a parabolic trajectory by a resistance proportional to velocity.

43. A small mass has initial velocity v_0 vertically upward at the point $s = 0$. Determine the fraction of the original kinetic energy that has been dissipated when it returns to the same position: (*a*) when resistance varies as speed; (*b*) when it varies as the square of speed.

44. A small mass has initial velocity v_0 vertically upward at a given point, and t_1 is the time that elapses until it returns. Prove that a resistance varying as v^2 shortens t_1 in comparison with the value for weight alone. Can the effects upon t_1 of buoyancy and such a resistance be made to compensate each other?

45. In the problem of § 119, assume that the resistance constant is influenced by the density of the medium and the horizontal cross-section area of the moving body. Then compare terminal velocities for spheres of different sizes and the same material in the same medium.

46. A body whose weight is w lies on the platform of an elevator that is moving vertically with acceleration p. What is the stress between the platform and the body?

47. A rectangular prism has a motion of rectilinear translation on a smooth horizontal plane, under the influence of force normal to its front face and uniformly distributed on that area. What internal stresses are caused within the prism?

48. A rectangular prism slides parallel to its longest axis on a smooth horizontal plane. It is pulled in the line of its axis by cords that pass over smooth pins and are attached to masses that hang freely. Required the acceleration of the system, the tension in each cord, and the total stress across any section of the prism normal to its axis.

49. Five equal blocks, with equally rough contact surfaces, are piled in the form of a rectangular prism, which stands on a smooth horizontal plane. An assigned velocity is to be imparted to the pile without disturbing the arrangement of blocks, by force parallel to the plane and applied to one block. Find the shortest time in which this can be done, and the place where the force should be exerted.

50. The centre C of a sphere is constrained to move along a smooth vertical guide that bisects the horizontal line joining two smooth pins A and B. Two unequal masses hang by cords that pass over the pins and are so attached to the sphere as to pull radially. Motion is caused by weight from a position of rest, C being below AB. Apply the work equation to find the velocities at any position. Calculate the tension in each cord and the acceleration of each mass.

51. A smooth horizontal guide is tangent to the circular section of a smooth cylinder-quadrant whose axis is horizontal. A mass is pulled along the guide by a heavy flexible rope that passes to the cylinder, over it in close contact, and then hangs vertically. Investigate the motion, the cross-section area of the rope being disregarded.

52. A given mass hangs by a given elastic cord. Find the period of an oscillation in the vertical line.

53. A smooth straight guide OA is horizontal, and can turn about a horizontal axis at O. A small mass m is threaded upon it. OA starts from rest and is raised above the horizontal by rotation about the axis with constant angular acceleration. Will m begin to move toward O or toward A? Will the result be modified if there is friction between m and OA?

54. A given smooth cylinder is fixed with its axis horizontal. A small mass m slides on the convex side of its circular section from rest at the highest point, the tangential force being due to weight. Find the place at which m leaves the cylinder.

55. The plane of a circle is vertical, and AB is its vertical diameter. Prove that the times are equal in which small masses will describe chords passing through A or B, the tangential force being due to weight. The masses start from rest.

56. A small mass is sliding in contact with the concave side of a given smooth circle whose plane is vertical. It passes

the lowest point with speed v_0; discuss the possible types of subsequent motion, tangential force being due to weight alone.

57. A mass moves, parallel to a slope-line, up an inclined plane, being retarded by weight and friction proportional to normal stress. Required the initial speed in order that the mass may come to rest at the end of a given interval. How far will it move during the interval?

58. A mass is moved parallel to a slope-line on an inclined plane, under the influence of weight, a force P of given magnitude, and friction with constant coefficient ϕ. At what angle with the slope-line must P act in order to produce maximum acceleration up the plane?

59. Establish the equations of motion and work when acceleration down the slope-line is produced by weight, a force P, and constant friction.

60. Prove that the resultant of normal stress and friction is vertical when a body slides by its weight down a plane inclined at the limiting angle.

61. Two planes of different inclination intersect in a horizontal line. Two masses slide on them, each parallel to a slope-line, and are joined by a cord so arranged that its parts are in the same vertical plane and parallel to the inclined planes. Apply the equations of motion and work to this case and discuss them, assuming friction proportional to normal stress to be active.

62. Two small masses are connected by a cord, and slide in a smooth horizontal tube that rotates with constant angular velocity about a vertical line intersecting its axis. If the radial velocity of both masses is initially zero, show that the cord will always remain taut, and calculate the stress in it at any position.

63. A small mass is attached by a cord to an axis, and makes complete revolutions at a given distance from it in a vertical plane. Compare the maximum and minimum values of angular velocity; of tension in the cord. How would the problem differ if the small mass were replaced by a sphere?

64. A smooth wire in the form of a circle is made to revolve uniformly about a vertical diameter. A small mass is threaded on the wire, and is initially at one end of a horizontal diameter, its vertical velocity being zero. If it falls along the wire as guide, will the mass reach the rotation-axis?

65. In the problem of § 125, express r as a function of t.

66. In the scheme of Fig. 31, p. 212, there is a hinge at A. BA is prolonged to C, and perpendicularity of OA and AB is secured by a brace OC, which breaks under a given stress. At what position of the sliding mass on AB will the break occur?

67. In connection with § 124, investigate the possibility of oscillation in the revolving tube for m_2 under those conditions. Also when m_1 is removed, the cord being elastic and fastened at A.

68. Two such arms as AC, Fig. 36, p. 230, are attached symmetrically to the axis, and revolve inside a concentric fixed cylinder, so that the contrivance acts as a brake. For given constant value of ω sufficient to produce stress between cylinder and spheres, calculate the power absorbed by the brake.

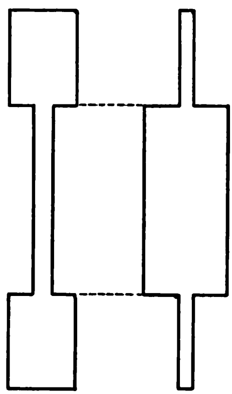
69. Water is pumped from a well of constant water-level to an (average) height h above the water-surface. Express the ratio of the power usefully applied and the total power for a given diameter of delivery-pipe.

70. Prove that the aggregate work of any three oblique component forces is equal to the work of their resultant for any displacement.

71. Two concentric cylindrical pulleys are free to rotate, as a rigid body, about their common geometrical axis, which is horizontal. Two masses are so hung by cords, one from the circumference of each cylinder, that one mass is pulled up when the other falls. Express the angular acceleration of the cylinders and the tensions in the cords. When the angular velocity is ω_1, the cord of the descending mass is cut; determine how far the other will rise before coming to rest.

72. Five equal disks with equally rough contact-surfaces are piled in the form of a circular cylinder that stands on a smooth horizontal plane. An assigned angular velocity is to be imparted without disturbing the pile, by a couple whose axis is parallel to the cylinder-axis, applied to one disk. Find the shortest time in which this can be done, and the disk to which the couple should be applied.

73. A cylinder is maintained in uniform rotation about its own axis which is fixed and vertical. Its upper base supports a concentric cylinder of equal radius with frictional contact, stress being due to weight and uniformly distributed over the bearing-surface. If the second cylinder has zero angular velocity initially, and friction is proportional to normal stress, at what time will the cylinders begin to rotate as one body?



A B

FIG. 41

74. Compare the times of rolling from rest down the slope-line of a given inclined plane for the compound cylindrical bodies A and B (Fig. 41). The diagram shows sections through the axes; the cylinders are of equal mass and external diameter; and equal radial distances are shown by the dotted lines.

75. An engine drives a machine steadily at h horse-power and n strokes per minute. A fly-wheel on the crank-shaft is to be so designed as to receive the whole work of one stroke with a given fractional change of angular velocity. Consider moment of inertia for the wheel-rim only.

76. Establish the relation between the moment for any axis of a resultant force and the moments of its oblique components.

77. A sphere rests in the angle between the horizontal XZ plane and the vertical YZ plane, with its centre in XY. In the plane last named a force is applied tangentially so as to increase the stress between sphere and planes at both contacts. If the coefficients of friction are equal, find the equation of condition that the sphere may turn.

78. A sphere and a cylinder are constrained to roll down an inclined plane, their centres moving along the same parallel to a line of slope. If they are initially at rest and in contact, which must be placed in front in order that the contact may continue? Suppose that arrangement to be made, with no friction between sphere and cylinder, and express the common acceleration of their centres. Write the equation of motion for each body and discuss it, if there is friction proportional to normal stress between the two.

79. A cylindrical pulley is composed of two semi-cylinders of different materials, joined on a diametral section. The compound cylinder turns about its geometrical axis, which is horizontal, and a mass m is hung by a cord that unwinds from the circumference of the pulley. What limiting value must m have in order that the cylinder may make complete revolutions, starting with its centre of mass in its lowest position? Calculate the period of a small oscillation about the position of equilibrium when m is less than this critical value.

80. Investigate whether the piston-speed is greatest when the axis of the pitman is tangent to the circle described by the centre of the crank-pin.

81. In the problem of § 133 are there any points of the prism whose paths are circles or ellipses? What is the path of its centre of mass? Can this become a straight line for any particular ratio of dimensions in the prism?

82. Four forces acting on a body are represented in magnitude, direction, and position by the sides of a gauche quadrilateral. Examine the condition that the force of their resultant may vanish; and locate the axis of its couple-moment when that condition is fulfilled.

83. A cylindrical shaft with axis vertical supports a given load, its end being fitted into a bearing which is a truncated cone. Supposing that the pressure per unit area is constant over the bearing-surface, and that friction is proportional to normal stress, find the power absorbed by friction for a given constant angular velocity of the shaft.

84. A belt carrying a scale-pan at each end hangs over a horizontal shaft that is made to turn with constant angular velocity by a motor. The scale-pans are so loaded that the belt hangs in equilibrium. What power is absorbed by the friction between belt and shaft?

85. A belt passes over two equal pulleys, and is to transmit h horse-power at n revolutions per minute to one of them without slip. What conditions determine the belt-tensions in their relation to the coefficient of friction between belt and pulley-face?

86. In the problem of § 123 calculate the smallest value for the coefficient of friction that is consistent with no slip of the cord. Discuss the motion of the two masses and the pulley when ϕ is less than this limiting value.

87. In connection with § 134 calculate the value of ϕ at a given inclination, required to produce a given ratio between ω and v ($\omega r_1 < v$).

88. A sphere that is rotating positively about a horizontal diameter parallel to Z is given in addition a positive translation parallel to X in contact with a horizontal plane. Investigate the subsequent motion as influenced by friction, especially with regard to a possible reversal of the translation and establishment of rolling.

89. A train is being pulled up a uniform grade at steady speed on a straight track, frictional resistance being a known fraction of weight for the engine and each car. Calculate the power that is necessary. Explain the scheme of external forces for the whole train and for each car separately.

90. A cylinder whose centre of mass is at C (Fig. 42) is caused to rotate positively at a constant rate about its own axis, which is fixed and horizontal, by a couple whose plane is that of the diagram. There is friction along a generator between the cylinder and a plane shown at AB. Prove that for a certain position of the element of contact Q the constraint at the bearings of the axis may be removed without affecting the motion. When there is loose fit between shaft and journal, some such action as this may take place; the shaft "climbs" on one side.

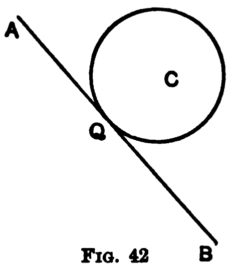


FIG. 42

91. A rectangular prism is placed at rest with its longest axis and one pair of faces horizontal, and is free to turn about a horizontal fixed axis through the centre of one base. As it rotates under the influence of weight, the horizontal and vertical forces of constraint exerted by the rotation-axis will vary.

Plot two curves showing each force as a function of angular displacement. Determine the maximum values and the positions at which they occur.

92. A rectangular prism slides with uniplanar motion inside a smooth fixed hemisphere whose base is uppermost and horizontal. Discuss the motion that takes place under the influence of weight, the guide-plane being vertical, and the prism at rest initially.

93. The hammer of a pile-driver has mass m_1 and falls freely through a height h before striking. The mass of the pile is m, the average resistance to penetration offered by the soil is R, and the coefficient of restitution e. Neglecting the effects of a second blow after rebound of the hammer, find how far the pile is driven.

94. In pile-driving is there greater efficiency in using a light hammer with long drop, or heavy hammer with shorter fall, the product of hammer-weight and vertical height being constant?

95. A car passes from a straight track to a curve, the speed remaining constant. What kinematical changes are produced in its motion, and in what general way are the dynamical effects brought about?

96. A mass m is moving in a straight line with velocity v_1. It is desired to change this to v_2 in t_1 seconds. Can the result be accomplished by pulling with a cord that breaks under a stress of n units?

97. A given total mass of steel is made into two spheres. For central impact between them with given velocities, required the ratio of their radii when the kinetic energy dissipated is a maximum.

98. A sphere falls vertically from rest at a height h above the horizontal surface of a large mass, and rebounds to a height

h_1. Determine from these data the coefficient of restitution for the substances concerned.

99. A rectangular prism is hinged at one edge to a horizontal platform on which it stands. Platform and prism have a horizontal motion of translation together perpendicular to the axis of the hinge. Required the conditions in order that the prism may just overturn, if the platform is suddenly reduced to rest.

100. When a water-jet impinges upon a horizontal plate (§ 129), how is the pressure changed if the plate has a vertical velocity $\pm v_1$?

101. Two given spheres are placed at rest with centres at a known distance. Find the time required for them to be brought into contact by their gravitational attraction alone, and the speed with which each is moving when they strike.

102. Examine the particular values of the final velocities for central impact, if the masses are equal, and e can be assumed as unity. This result has important bearings upon some problems of molecular physics.

103. What is the condition that one sphere should be reduced to rest by central impact upon another that is initially at rest?

104. The velocities of two spheres being equal and opposite, find the condition that one of them should be left at rest after central impact.

105. Two spheres m_1, m_2, are at rest; a third moving on the line of centres with velocity v strikes m_1, which in turn strikes m_2. Determine m_1 so that the velocity communicated to m_2 shall be a maximum, other external forces being excluded.

106. A plank that is free to move is perforated on a line through its centre of mass by a bullet. Assuming that the

resistance offered to the latter is constant at all speeds, prove that the momentum imparted to the plank is smaller, the greater the speed of the bullet.

107. Wind blows steadily parallel to the axis of a windmill that turns uniformly. The arms are plane, and it is required to express the distribution of pressure per unit area along their radial central lines. The pressure is assumed proportional to the square of the wind's velocity relative to the arm, reckoned in a normal to the latter.

108. A rectangular prism lies at rest on a smooth horizontal plane, on which it can slide freely. It begins to rotate under the impulse of a horizontal force whose direction intersects the longest prism-axis at right angles. The time-interval being so short that changes of configuration may be disregarded, determine the position of the force applied, in order that the initial rotation-axis shall pass through a given point of the prism-axis.

109. Two equal rectangular prisms are suspended by axes that are similar median lines of their upper faces. The axes are parallel and in the same horizontal plane; the centres of the prisms lie in the same vertical plane perpendicular to the axes. They are perforated in succession by a bullet which can be considered as moving in that vertical plane and horizontally during the time of its action. Compare the moments of momentum communicated to the two prisms, supposing the same resistance to penetration for both.

110. Two equal spheres of different mass hang by parallel equal cords from a horizontal plane, and are just in contact. One centre is drawn aside through a given angle in the plane of the cords and released. The angle is observed through which the other cord has moved when the centre of that sphere comes to rest. How can the coefficient of restitution be calculated by this method?

111. In the problem of § 132 the momentum mv is given. Find the value of h for which maximum angular velocity will be produced in the other body.

112. A vertical spindle OQ (Fig. 43) revolves in fixed bearings, and carries a frame $ABCD$, attached at D and hinged at A, B, C, D, so as to allow motion in a vertical plane. The four arms are equal, and carry equal spheres A and B with centres at those angles. A third sphere C, with centre at the lowest angle, slides without friction on OQ. Find the angular velocity ω corresponding to a given configuration of steady motion. How much work will be done by the weight of the spheres in passing to a new adjustment for steady motion when the change in ω is known?

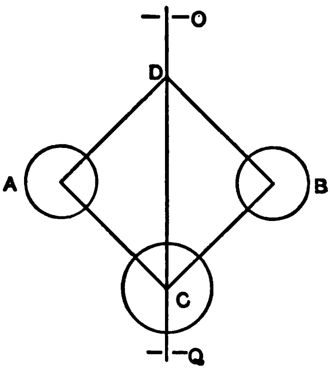


Fig. 43

113. Express the acceleration of m perpendicular to the guide AB (Fig. 31, p. 212) if it is curved in the plane of the diagram.

114. The mass m (in the same figure) is replaced by a sphere which is constrained to roll along the front face of the revolving straight guide. Investigate the energy relations.

115. The object of raising the outer rail on a curved track is to furnish the required constraint as the resultant of weight and reaction normal to the track. Deduce the equation of condition for a given average speed on a curve of known radius.

116. A pendulum beats in t' seconds at lat. 45° N. and sea-level in a vacuum. Express the weight in dynes and in pounds-

weight of m grams at a place where the same pendulum beats in one second.

117. Deduce the approximate rate of variation in g along the same vertical, the distance from the earth's surface being small in comparison with its radius.

118. Two pendulums that can be regarded as simple have rates $\pm n$ seconds a day at a given locality, n being a small number, and lengths l_1, l_2. Calculate the length of the simple pendulum beating seconds at the same place.

119. The pendulum of a clock consists essentially of a glass jar containing mercury and attached to a steel rod. Explain how the time of beat for such a pendulum may be rendered independent of changes in temperature.

120. In the case of a weight pendulum, plot curves showing the velocity and the tangential acceleration of its centre of mass as functions of angular displacement.

121. A small mass is constrained to vibrate (without tangential resistance) about the lowest point of any curve in a vertical plane, under the influence of weight. On what basis can the approximate period of a small oscillation be expressed?

122. Show that one element determining period in the various types of pendulum can be analyzed as force-moment per unit angle of displacement from a definite position.

123. A cylinder oscillates as a weight pendulum about a horizontal line intersecting its geometrical axis at right angles. Find the position of the rotation-axis for which the period is a minimum.

124. A weight pendulum vibrating about a horizontal axis at O consists of a rod whose mass may be disregarded, and a sphere with centre at C. Two spheres of equal mass, which

is a small fraction of the total, are to be attached with their centres in the line OC (prolonged if necessary) without changing the original period. How can this be done?

125. A pendulum is composed of a cylinder and an adjustable slider in the form of a concentric hollow cylinder. Express the rate of change in the period produced by a small displacement of the slider at any position. Is there an adjustment at which this rate passes through the value zero?

126. A pendulum consists of a homogeneous cylinder capped with two homogeneous hemispheres, equal in volume but with masses m_1, m_2, their centres being on the cylinder-axis. The radius of the hemispheres is to be so chosen that the altitude of the cylinder is the length of the equivalent simple pendulum at a given locality.

127. A cylinder is fixed with its axis horizontal, and a cylindrical disk rolls inside it under the influence of weight, the generators of the two cylinders remaining parallel. Regard the disk as a weight pendulum, and find the period of its centre.

128. A sphere supported by a wire of negligible mass is executing small vibrations as a weight pendulum against resistance kv^2, k being small. Prove that the period is not altered by the resistance, and establish the law of diminution for amplitude.

129. The equilibrium position for a damped oscillation (§ 91) is to be deduced from observation of three successive turning-points on a fixed scale in the line of motion. The damping-factor has any value consistent with real vibration.

130. Find the correction for amplitude to be made, when the time has been observed that is required for n beats of damped oscillation in a circular pendulum. Observe that the amplitudes form a geometrical series (eq. (8), § 92), while the general form for small correction is given in equation (7), § 96.

131. Prove that if three (partial resultant) forces produce equilibrium, their lines of action must intersect in one point.

132. Prove that a couple-moment cannot be the equilibrant of a resultant that includes a force-term.

133. A paraboloid of revolution rests on a horizontal plane. Determine the positions of possible equilibrium under the influence of weight and the reaction of the plane.

134. A hemisphere stands with its convex surface on a horizontal plane, and a mass m is hung by a cord from a point in its rim. Locate the point of contact with the plane for equilibrium.

135. A square board is hung flat against a wall by means of a cord attached to the extremities of its upper edge and passing over a smooth pin. Prove that if the length of the cord is less than the diagonal of the board, there will be three positions of equilibrium.

136. A given window-sash slides with some play in its guides, and is balanced by equal sash-weights. One cord breaks, and the sash is just held by friction; discuss the conditions.

137. A sphere rests between plane jaws, their bisector being vertical upward and their line of intersection horizontal. One jaw is fixed; the other is hinged to it, and has a horizontal arm with a movable counterpoise, whose weight tends to bring the jaws together. Required the equation of condition for equilibrium when there is no friction on the sphere. Also the range of the counterpoise between incipient motion of the sphere upward and downward, when there is friction between the jaws and the sphere.

138. In connection with § 139 examine whether there is a limiting value of ϕ beyond which the prism will stand in any position without slipping.

139. When a body is flexible, like a rope, the force-equation for equilibrium is sufficient, and the moment-equation is not required. Show this to be true.

140. A uniform heavy rope hangs in the form of a catenary over two smooth pins, being held in equilibrium by parts that hang vertically. Prove that the lower ends of the rope are in a horizontal line.

141. Two equal bars, AB and AC, are hinged at A, their median sections being in the same vertical plane, and the hinge-axis horizontal. Their lower ends, B and C, slide without friction along a straight slot in a horizontal plane. A mass hangs by a cord from A, and a heavy rope in the form of a catenary from B and C. Determine the conditions of equilibrium.

142. When the motion is elliptical harmonic, the speed at any position varies as the length of the diameter conjugate to one drawn through the point in question.

143. It has been shown (§ 111) that the major-axes are equal for all ellipses that satisfy a certain condition. What element determines the minor-axes of those ellipses, and how is the relation expressed quantitatively?

144. Deduce the periodic time in the ellipse from equation (5), § 111.

145. Find an expression giving the maximum value of v_r, for an ellipse described as a central orbit round a focus, in terms of the major-axis, period, and eccentricity.

146. Express the major-axis and period for the orbit of the moon relative to an origin at the centre of the earth (system (F_1), § 104).

147. Work out in its essential elements a scheme for determining the mass of the moon.

148. Decompose the element of difference between G and $\hat{g}$ (§ 106) into the parts which introduce: (*a*) divergence of direction; (*b*) variation in magnitude.

149. If an ellipse is described as a central orbit round a focus, the velocity at any point can be decomposed into two components of constant magnitude, perpendicular respectively to the radius-vector and the major-axis.

150. Calculate the moment of inertia of a rectangular prism: (*a*) for an axis through the centre of mass parallel to one edge; (*b*) for a diagonal of a median section; (*c*) for a diagonal of the prism.

151. Calculate the moment of inertia of an anchor ring for its axis of symmetry.

152. Establish a relation between products of inertia for any set of rectangular axes, and those for parallel axes at the centre of mass.

153. Investigate the limiting form of the momental ellipsoid at the centre of a cylinder when the radius or the altitude approaches zero.

154. "The astronomical unit of mass is that mass which attracts another body placed at the unit of distance so as to produce in that body the unit of acceleration. If, as in the astronomical system, the unit of mass is defined with respect to its attractive power, the dimensions of $[M]$ are $[L]^3\,[T]^{-2}$" (Maxwell). Discuss this statement in connection with §§ 142–3. Calculate the mass of the earth in astronomical units.

155. Compare the results of the discussion in § 106 with the statement: "It is shown by theory, and verified by observation, that the variation in g (at sea-level) is proportional to the square of the sine of the latitude."

156. A rectangular block slides parallel to a slope-line down the smooth inclined face of a wedge that is itself free to move over a smooth horizontal plane. The system is initially at rest; apply the three fundamental equations to this case. Required the minimum coefficient of friction between wedge and horizontal plane, in order to hold the former at rest while the block slides down it.

157. Apply the fundamental equations to the problem of § 134, modified so that the inclined plane is itself free to move on a smooth horizontal plane. Suppose the constraint that produces rolling of the cylinder to be exercised by a cord, and investigate especially the tension in it.

158. A sphere is constrained to roll along a line of slope on an inclined plane. A cord is always parallel to the slope-line, and tangent to the sphere at the upper end of the diameter through the point of contact. It unwinds for motion of the sphere up the plane, passes over a smooth pin, and carries a given mass hanging freely. Express the acceleration of the sphere's centre. Discuss the possible (uniplanar) motions if there is limited constraint by friction between sphere and plane.

159. A cylinder is rigidly attached to a vertical rotation-axis that passes through its centre and makes an angle β with the cylinder-axis. Determine the directive couple-moment which the axis supplies when the angular velocity is given.

160. Write equation (5), § 137, in the form

$$-m_1 g\bar{r} = -\omega^2 \cos\gamma\,(I_{Y_1} - I_{X_1}),$$

plot γ as a function of ω, and interpret the conditions exhibited by the curve.

161. A rectangular prism turns about a horizontal line through a point of its longest axis and parallel to an edge.

The prism falls from rest by its weight through an angle α to a position where its axis is horizontal, and is brought to rest in one-twentieth of a second by a pin placed vertically in its median plane. Calculate the average stress between prism and pin for the interval as a function of the pin's position.

162. In connection with the centre of mass as the centroid of a homogeneous body, prove the following (geometrical) propositions:

(I) The volume generated by the rotation of any plane area about a line in its plane that does not cut through it is equal to the product of the area and the circumference described by its centroid.

(II) The area generated by the rotation of any plane curve about a line in its plane that does not cut through it is equal to the product of the length of the curve and the circumference described by its centroid.

These are known as "Guldin's Rules."

163. Investigate rectilinear motion under the influence of central force proportional to displacement and resistance of magnitude $(k_1 v + k_2 v^2)$.

APPENDIX

The critical discussion of the Laws of Motion, as they were announced by Newton, is beginning to bear fruit. The time has fairly come when those laws can be restated in terms that conform more closely to our present view of the conceptions really involved, and of their logical relation to each other. The Newtonian forms of statement have therefore not been introduced into the text, in the belief that the student who is entering upon the subject of Mechanics needs a presentation of fundamental matters that shows the clearest thought concerning them to which we have thus far attained. But tradition demands that the historical words of the Principia should at least be recognized somewhere and quoted; so they are given below in an accepted English version. It is quite possible, or even to be expected, that a first reading of them will show no reason why any attempt should be made to supersede them. If interest is awakened on these points, no better book is available as a guide to inquiry than the "Science of Mechanics, a Critical and Historical Exposition of its Principles," by Professor Ernst Mach, of Vienna. It is published in English translation.

Newton's Laws of Motion

Law I. Every body continues in its state of rest or of uniform motion in a straight line, except in so far as it may be compelled by impressed forces to change that state.

Law II. Change of motion is proportional to the impressed force, and takes place in the direction of the straight line in which the force acts.

Law III. To every action there is always an equal and contrary reaction; or, the mutual actions of any two bodies are always equal and oppositely directed in the same straight line.

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